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Isospectral Flat Connexions

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ABSTRACT: Isospectral flat connexions are constructed for higher rank bundles over compact Riemann surfaces

Key Words: Flat connexions ; Spectrum; compact Riemann surfaces.

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1. Isospectral Flat Connexions

We use R.Kuwabara's construction [3] of isospectral flat line bundles over certain compact Riemann surfaces to show similar examples in rank n. This gives distinct unitary representations of a Fuchsian group with the same Selberg Zeta function (since the spectrum determines the Zeta function via the length spectrum([1], [2], [5])).

Let X be a compact Riemann surface of genus $g > 1$. $X$is a quotient of the upper half plane $H = \{x+iy|y > 0\}$ by a subgroup Γ of $SL_2(R)$ under the Moebius action. Γ is isomorphic to the fundamental group of X, $\pi_1(X)$. Let $Pic_n(X)$ be the set of equivalence classes of irreducible unitary representations

$$\chi : \pi_1(X) \to U(n)$$

. Using the values of χ for (locally constant)transition functions,one has an indecomposable flat holomorphic vector bundle $E_\chi \to X$([4, Sec4]).

Δ_H denotes the Laplace - Beltrami operator in the upper half plane.

$$\Delta_H = -y^2(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2})$$

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Let $C^\infty(\Gamma, \chi)$ be the space of C^∞ functions f with values in C^n and equivariant for (Γ, χ)

i.e. $f(\gamma\chi) = \chi(\gamma)f(x)$ for $x \in H$, $\gamma \in \Gamma$.

Δ_H acts on $C^\infty(\Gamma, \chi)$ by the natural diagonal extension. Δ_H maps $C^\infty(\Gamma, \chi)$ to itself and the restriction map is denoted by Δ_χ. Elements of $C^\infty(\Gamma, \chi))$ correspond

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naturally to smooth sections of E_χ. Under this correspondence Δ_χ is identified with the Laplacian of the flat connexion on E_χ.

The determinant map gives for each unitary representation χ, a unitary character $det\chi$. Now the determinant bundle of E_χ, $detE_\chi$ is the same as $E_{det\chi}$ since the transition functions of the two bundles are the same. Also there is a natural map $f \to \tilde{f}$ from $C^\infty(\Gamma, \chi))$ to $C^\infty(\Gamma, det\chi)$ given by multiplying the coordinate functions of $f = (f_1, f_2, ..., f_n)$. If f is an eigenfunction of Δ_χ for eigenvalue λ $\lambda \geq 0$, then $\tilde{f}$ is an eigenfunction of $\Delta_{det\chi}$ for eigenvalue λ^n. Thus the assignment $\lambda \mapsto \lambda^n$ gives an injection from Spectrum Δ_χ into Spectrum $\Delta_{det\chi}$.
Equivalent representations have identical determinants.

Proposition 1.1. *There exist inequivalent irreducible representations* $\chi_1, \chi_2 :$ $\pi_1(X) \to U(n)$ *with spectrum* $\Delta_{\chi_1}=$ *spectrum* Δ_{χ_2} *(for suitable* X*).*

Proof: Let $\rho_1, \rho_2 : \pi_1(X) \to S^1 = U(1)$ denote Kuwabara's examples [3, p 472]. Observe that the range of ρ_1,ρ_2 is $\{+1, -1, i, -i\}$. Recall that $\pi_1(X)$ has a presentation $\{a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i] = 1\}$. Now in Kuwabara's construction $b_i \mapsto 1$ for each $i = 1, 2, ..., g$. Using 2 x 2 blocks of rotation matrices one constructs irreducible sets of n x n unitary matrices $\{A_1, ..., A_g\}$ and $\{A_1^{'}, ..., A_g^{'}\}$such that

$$detA_i = \rho_1(a_i), \quad detA_i^{'} = \rho_2(a_i) \qquad (i = 1, 2, ..., g)$$

Set $\chi_1(a_i) = A_i$, $\chi_2(a_i) = A_i^{'}$, $\chi_2(b_i) = \chi_1(b_i)$= identity (i=1,2,...,g). Since spectrum Δ_{ρ_1} = spectrum Δ_{ρ_2}, one has spectrum Δ_{χ_1} = spectrum Δ_{χ_2} . Now $\rho_1 = det_{\chi_1}, \rho_2 = det_{\chi_2}$ and ρ_1 is not equivalent to ρ_2.

Hence χ_1 is not equivalent to χ_2, and χ_1, χ_2 give distinct points of $Pic_n(X)$. □

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