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Construction of Inextensible Flows of Dual Normal Surfaces in the Dual Space $\mathbb{D}^3$

Vedat Asil and Talat Körpınar

ABSTRACT: In this paper, we study dual normal surfaces in dual space $\mathbb{D}^3$. We research inextensible flows of dual normal surfaces of dual curves in dual space $\mathbb{D}^3$.

Key Words: Dual space curve, inextensible flows, Structural mechanics, Dual Frenet frame.

Contents

1. Introduction

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

This study is organised as follows: Firstly, we study dual normal surfaces in dual space $\mathbb{D}^3$. Finally, we research inextensible flows of dual normal surfaces of dual curves in dual space $\mathbb{D}^3$.

2. Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

There is a tight connection between spatial kinematics and the geometry of line in the three-dimensional Euclidean space $\mathbb{E}^3$ Therefore we start with recalling the use of appropriate line coordinates: An oriented line L in the three threedimensional Euclidean space $\mathbb{E}^3$ can be determined by a point $p \in L$ and a normalized direction vector x of L,i.e. $\|x\| = 1$.To obtain components for L [2], one forms the moment vector

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$$x^* = p \times x, \tag{2.1}$$

with respect to the origin point in $\mathbb{E}^3$.If p is substituted by any point

$$q = p + \mu x, \mu \in \mathbb{R}, \tag{2.2}$$

on L, Eq. (2.1) implies that $x^*$is independent of p on L.The two vectors x and x^* are not independent of one another; they satisfy the following relationships:

$$\langle x, x \rangle = 1, \quad \langle x^*, x \rangle = 0. \tag{2.3}$$

The six components x_i and x_i^* $(i = 1, 2, 3)$ of x and x^* are called the normalized Plücker coordinates of the line L. Hence the two vectors x and x^* determine the oriented line L.

Conversely, any six-tuple x_i , $x_i^*(i = 1, 2, 3)$ with

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0, \tag{2.4}$$

represents a line in the three-dimensional Euclidean space $\mathbb{E}^3$. Thus, the set of all oriented lines in the three-dimensional Euclidean space $\mathbb{E}^3$ is in one-to-one correspondence with pairs of vectors in $\mathbb{E}^3$ subject to the relationships in Eq. (2.3).

For all pairs $(x, x^*) \in \mathbb{E}^3 \times \mathbb{E}^3$ the set

$$\mathbb{D}^3 = \left\{ X = x + \varepsilon x^*, \varepsilon \neq 0, \varepsilon^2 = 0 \right\}, \tag{2.5}$$

together with the scalar product

$$\langle X, Y \rangle = \langle x, y \rangle + \varepsilon \left(\langle y, x^* \rangle + \langle y^*, x \rangle \right), \tag{2.6}$$

forms the dual 3-space $\mathbb{D}^3$, [2]. Thereby a point $X = (X_1 + X_2 + X_3)^t$ has dual coordinates $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. The norm is defined by

$$\langle X, X \rangle^{\frac{1}{2}} = \|X\| = \|x\| \, (1 + \varepsilon \frac{\langle x, x^* \rangle}{\|x\|^2}). \tag{2.7}$$

In the dual 3-space $\mathbb{D}^3$ the dual unit sphere is defined by

$$K = \left\{ A \in \mathbb{D}^3 : \|X\|^2 = X_1^2 + X_2^2 + X_3^2 = 1 \right\}. \tag{2.8}$$

The set of all oriented lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual 3-space $\mathbb{D}^3$. The representation of directed lines in $\mathbb{E}^3$ by dual unit vectors brings about several advantages and from now on we do not distinguish between oriented lines and their representing dual unit vectors.

Let $\left\{\hat{\mathbf{T}},\hat{\mathbf{N}},\hat{\mathbf{B}}\right\}$ be the dual Frenet frame of the differentiable dual curve in the dual space $\mathbb{D}^3$. Then the dual Frenet frame equations are

$$\begin{aligned}
\hat{\mathbf{T}}' &= \kappa\mathbf{N}+\varepsilon(\kappa^*\mathbf{N}+\kappa\mathbf{N}^*),\\
\mathbf{N}' &= -\kappa\mathbf{T}+\tau\mathbf{B}+\varepsilon\left(-\kappa^*\mathbf{T}-\kappa\mathbf{T}^*+\tau^*\mathbf{B}+\tau\mathbf{B}^*\right),\\
\hat{\mathbf{B}}' &= -\tau\mathbf{N}-\varepsilon\left(\tau^*\mathbf{N}+\tau\mathbf{N}^*\right),
\end{aligned} \tag{2.9}$$

where $\hat{\kappa}=\kappa+\varepsilon\kappa^*$ is nowhere pure dual natural curvatures and $\hat{\tau}=\tau+\varepsilon\tau^*$ is nowhere pure dual torsion

3. Inextensible Flows of Dual Normal Surfaces in the $\mathbb{D}^3$

The dual normal surface of $\hat{\gamma}$ is a ruled surface

$$\hat{\mathcal{M}}\left(s,u\right)=\gamma\left(s\right)+u\mathbf{N}\left(s\right)+\varepsilon(\gamma^*\left(s\right)+u\mathbf{N}^*\left(s\right)).$$

Let $\hat{\varpi}$ be the dual standard unit normal vector field on a dual surface $\hat{\mathcal{M}}$ defined by

$$\hat{\varpi}=\frac{\hat{\mathcal{M}}_s\wedge\hat{\mathcal{M}}_u}{\left\|\hat{\mathcal{M}}_s\wedge\hat{\mathcal{M}}_u\right\|}.$$

Then, the first fundamental form $\widehat{\mathbf{I}}$ and the second fundamental form $\widehat{\mathbf{II}}$ of a surface $\hat{\mathcal{M}}$ are defined by, respectively,

$$\begin{aligned}
\widehat{\mathbf{I}} &= \hat{\mathbf{E}}ds^2+2\hat{\mathbf{F}}dsdu+\hat{\mathbf{G}}du^2,\\
\widehat{\mathbf{II}} &= \hat{\mathbf{e}}ds^2+2\hat{\mathbf{f}}dsdu+\hat{\mathbf{g}}du^2,
\end{aligned}$$

where

$$\hat{\mathbf{E}}=\left\langle\hat{\mathcal{M}}_s,\hat{\mathcal{M}}_s\right\rangle,\quad \hat{\mathbf{F}}=\left\langle\hat{\mathcal{M}}_s,\hat{\mathcal{M}}_u\right\rangle,\quad \hat{\mathbf{G}}=\left\langle\hat{\mathcal{M}}_u,\hat{\mathcal{M}}_u\right\rangle,$$

$$\begin{aligned}
\hat{\mathbf{e}} &= -\left\langle\hat{\mathcal{M}}_s,\hat{\varpi}_s\right\rangle=\left\langle\hat{\mathcal{M}}_{ss},\hat{\varpi}\right\rangle,\\
\hat{\mathbf{f}} &= -\left\langle\hat{\mathcal{M}}_s,\hat{\varpi}_u\right\rangle=\left\langle\hat{\mathcal{M}}_{su},\hat{\varpi}\right\rangle,\\
\hat{\mathbf{g}} &= -\left\langle\hat{\mathcal{M}}_u,\hat{\varpi}_u\right\rangle=\left\langle\hat{\mathcal{M}}_{uu},\hat{\varpi}\right\rangle.
\end{aligned} \tag{3.1}$$

On the other hand, the dual Gaussian curvature $\hat{\mathbf{K}}$ and the dual mean curvature $\hat{\mathbf{H}}$ are

$$\hat{\mathbf{K}}=\frac{\hat{\mathbf{e}}\hat{\mathbf{g}}-\hat{\mathbf{f}}^2}{\hat{\mathbf{E}}\hat{\mathbf{G}}-\hat{\mathbf{F}}^2}, \tag{3.2}$$

$$\hat{\mathbf{H}}=\frac{\hat{\mathbf{E}}\hat{\mathbf{g}}-\mathbf{2}\hat{\mathbf{F}}\hat{\mathbf{f}}+\hat{\mathbf{G}}\hat{\mathbf{e}}}{2\left(\hat{\mathbf{E}}\hat{\mathbf{G}}-\hat{\mathbf{F}}^2\right)},$$

respectively.

Definition 3.1. *A surface evolution* $\hat{\mathcal{M}}(s,u,t)$ *and its flow* $\frac{\partial \hat{\mathcal{M}}}{\partial t}$ *are said to be inextensible if its first fundamental form* $\{\hat{\mathbf{E}}, \hat{\mathbf{F}}, \hat{\mathbf{G}}\}$ *satisfies*

$$\frac{\partial \hat{\mathbf{E}}}{\partial t} = \frac{\partial \hat{\mathbf{F}}}{\partial t} = \frac{\partial \hat{\mathbf{G}}}{\partial t} = 0. \tag{3.3}$$

Definition 3.2. *We can define the following one-parameter family of dual normal surface*

$$\hat{\mathcal{M}}(s,u,t) = \gamma(s,t) + u\mathbf{N}(s,t) + \varepsilon(\gamma^*(s,t) + u\mathbf{N}^*(s,t)). \tag{3.4}$$

Theorem 3.3. *Let* $\hat{\mathcal{M}}$ *is the dual normal surface in* $\mathbb{D}^3$. *If* $\frac{\partial \hat{\mathcal{M}}}{\partial t}$ *is inextensible, then*

$$\begin{aligned} -2u\frac{\partial \kappa}{\partial t} + 2u^2(\kappa\frac{\partial \kappa}{\partial t} + \tau\frac{\partial \tau}{\partial t}) &= 0, \\ -2u\frac{\partial \kappa^*}{\partial t} + 2u^2\frac{\partial}{\partial t}(\kappa\kappa^* + \tau\tau^*) &= 0. \end{aligned} \tag{3.5}$$

Proof: Assume that $\hat{\mathcal{M}}(s,u,t)$ be a one-parameter family of dual normal surface.

$$\begin{aligned} \mathcal{M}_s(s,u,t) &= (1-u\kappa)\mathbf{T} + \tau u\mathbf{B}, \\ \mathcal{M}_s^*(s,u,t) &= \mathbf{T}^* - u\kappa\mathbf{T}^* - u\kappa^*\mathbf{T} + u\tau\mathbf{B}^* + u\mathbf{B}\tau^* \\ \mathcal{M}_u(s,u,t) &= \mathbf{N} \\ \mathcal{M}_u^*(s,u,t) &= \mathbf{N}^* \end{aligned}$$

If we compute first fundamental form $\{\hat{\mathbf{E}}, \hat{\mathbf{F}}, \hat{\mathbf{G}}\}$, we have

$$\begin{aligned} \mathbf{E} &= (1-u\kappa)^2 + u^2\tau^2 \\ \mathbf{E}^* &= -2u\kappa^* + 2u^2(\kappa\kappa^* + \tau\tau^*), \\ \mathbf{F} &= \mathbf{F}^* = 0, \\ \mathbf{G} &= 1, \ \mathbf{G}^* = 0. \end{aligned} \tag{3.6}$$

From above system, we have

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= -2u\frac{\partial \kappa}{\partial t} + 2u^2(\kappa\frac{\partial \kappa}{\partial t} + \tau\frac{\partial \tau}{\partial t}), \\ \frac{\partial \mathbf{E}^*}{\partial t} &= -2u\frac{\partial \kappa^*}{\partial t} + 2u^2\frac{\partial}{\partial t}(\kappa\kappa^* + \tau\tau^*), \\ \frac{\partial F}{\partial t} &= \frac{\partial F^*}{\partial t} = 0, \\ \frac{\partial G}{\partial t} &= \frac{\partial G^*}{\partial t} = 0. \end{aligned}$$

Therefore, $\dfrac{\partial \hat{\mathcal{M}}}{\partial t}$ is inextensible iff

$$\begin{aligned} -2u\frac{\partial \kappa}{\partial t} + 2u^2(\kappa\frac{\partial \kappa}{\partial t} + \tau\frac{\partial \tau}{\partial t}) &= 0, \\ -2u\frac{\partial \kappa^*}{\partial t} + 2u^2\frac{\partial}{\partial t}(\kappa\kappa^* + \tau\tau^*) &= 0. \end{aligned}$$

This concludes the proof of theorem.

□

Corollary 3.4. *Let $\hat{\mathcal{M}}$ is the dual normal surface in $\mathbb{D}^3$. If flow of this dual normal surface is inextensible, then this surface is minimal if*

$$\begin{gathered} \frac{C^3}{2}[u^2\frac{\partial \kappa}{\partial s}\tau + u\frac{\partial \tau}{\partial s} - u^2\frac{\partial \tau}{\partial s}\kappa] = 0, \\ \frac{C^3}{2}[u^2(\frac{\partial \kappa}{\partial s}\tau^* + \frac{\partial \kappa^*}{\partial s}\tau) + u\frac{\partial \tau^*}{\partial s} - u^2(\frac{\partial \tau}{\partial s}\kappa^* + \frac{\partial \tau^*}{\partial s}\kappa)] + \\ \frac{3C^2C^*}{2}[u^2\frac{\partial \kappa}{\partial s}\tau + u\frac{\partial \tau}{\partial s} - u^2\frac{\partial \tau}{\partial s}\kappa] = 0, \end{gathered} \tag{3.7}$$

where $\hat{\mathcal{N}} = \left[(1-u\kappa)^2 + u^2\tau^2 + \varepsilon(2\kappa^*(-2u+u^2\kappa)\right]^{\frac{1}{2}}$ *and* $\hat{C} = \frac{1}{\hat{\mathcal{N}}}$.

Proof: Using $\hat{\mathcal{M}}_s$ and $\hat{\mathcal{M}}_u$, we get

$$\begin{aligned} \mathcal{M}_{ss} &= -u\frac{\partial \kappa}{\partial s}\mathbf{T} + (\kappa - u\kappa^2 - u\tau^2)\mathbf{N} + u\frac{\partial \tau}{\partial s}\mathbf{B} \\ \mathcal{M}^*_{ss} &= \kappa^* - 2u(\kappa\kappa^* + \tau\tau^*)\mathbf{N} + (\kappa - u\kappa^2 - u\tau^2)\mathbf{N}^*, \\ \mathcal{M}_{su} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \mathcal{M}^*_{su} &= -\kappa\mathbf{T}^* - \kappa^*\mathbf{T} + \tau\mathbf{B}^* + \tau^*\mathbf{B}, \\ \mathcal{M}_{uu} &= \mathcal{M}^*_{uu} = 0. \end{aligned} \tag{3.8}$$

On the other hand, the dual standard unit normal vector field on a surface $\hat{\mathcal{M}}$ is

$$\begin{aligned} \varpi &= C[(1-u\kappa)\mathbf{B} - u\tau\mathbf{T}], \\ \varpi^* &= C[\mathbf{B}^* - u\kappa^*\mathbf{B} - u\kappa\mathbf{B}^*] + C^*[(1-u\kappa)\mathbf{B} - u\tau\mathbf{T}], \end{aligned}$$

where $\hat{\mathcal{N}} = \left[(1-u\kappa)^2 + u^2\tau^2 + \varepsilon(2\kappa^*(-2u+u^2\kappa)\right]^{\frac{1}{2}}$ and $\hat{C} = \frac{1}{\hat{\mathcal{N}}}$.

Components of second fundamental form of dual surface are

$$\begin{aligned}
\mathbf{e} &= C[u^2\frac{\partial \kappa}{\partial s}\tau + u\frac{\partial \tau}{\partial s} - u^2\frac{\partial \tau}{\partial s}\kappa], \\
\mathbf{e}^* &= C[u^2(\frac{\partial \kappa}{\partial s}\tau^* + \frac{\partial \kappa^*}{\partial s}\tau) + u\frac{\partial \tau^*}{\partial s} - u^2(\frac{\partial \tau}{\partial s}\kappa^* + \frac{\partial \tau^*}{\partial s}\kappa)] + \\
& \quad C^*[(u^2\frac{\partial \kappa}{\partial s}\tau + u\frac{\partial \tau}{\partial s} - u^2\frac{\partial \tau}{\partial s}\kappa)], \\
\mathbf{f} &= C\tau, \quad \mathbf{f}^* = C^*\tau + C\tau^*, \\
\mathbf{g} &= \mathbf{g}^* = 0.
\end{aligned} \tag{3.9}$$

Therefore, using above system and Eq.(3.9), we obtain

$$\begin{aligned}
\mathbf{H} &= \frac{C^3}{2}[u^2\frac{\partial \kappa}{\partial s}\tau + u\frac{\partial \tau}{\partial s} - u^2\frac{\partial \tau}{\partial s}\kappa], \\
\mathbf{H}^* &= \frac{C^3}{2}[u^2(\frac{\partial \kappa}{\partial s}\tau^* + \frac{\partial \kappa^*}{\partial s}\tau) + u\frac{\partial \tau^*}{\partial s} - u^2(\frac{\partial \tau}{\partial s}\kappa^* + \frac{\partial \tau^*}{\partial s}\kappa)] + \\
& \quad \frac{3C^2C^*}{2}[u^2\frac{\partial \kappa}{\partial s}\tau + u\frac{\partial \tau}{\partial s} - u^2\frac{\partial \tau}{\partial s}\kappa].
\end{aligned} \tag{3.10}$$

where $\hat{\mathcal{N}} = \left[(1-u\kappa)^2 + u^2\tau^2 + \varepsilon(2\kappa^*(-2u+u^2\kappa)\right]^{\frac{1}{2}}$ and $\hat{C} = \frac{1}{\hat{\mathcal{N}}}$.

By using the relations Eq.(3.9) and Eq.(3.10), we obtain Eq.(3.7). This completes the proof.

□

Hence, we have the following result without proof.

Corollary 3.5. *Let $\hat{\mathcal{M}}$ is the dual normal surface in $\mathbb{D}^3$. If γ is a dual helix, then this surface is minimal.*

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Vedat Asil
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: vasil@firat.edu.tr

and

Talat Körpınar
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: talatkorpinar@gmail.com