Bol. Soc. Paran. Mat. (3s.) **v. 31** 2 (2013): 273–277.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i2.18616

On Construction of $\mathcal{D}-$Focal Curves in Euclidean 3-Space $\mathbb{M}^3$

Vedat Asil, Selcuk Bas and Talat Körpinar

ABSTRACT: In this paper, we study $\mathcal{D}-$focal curves in the Euclidean 3-space $\mathbb{M}^3$. We characterize $\mathcal{D}-$focal curves in terms of their focal curvatures.

Key Words: Darboux frame, Euclidean 3-space, Focal curve, Normal curvature, Geodesic Curvature, Geodesic torsion.

Contents

1. Background on curves and surfaces

In the following discussion, all curves and surfaces are considered to be regular and "sufficiently smooth." A curve is regular if it admits a tangent line at each point, while a surface is regular if it admits a tangent plane at each point. All surfaces are considered to be *oriented*. A surface is said to be oriented if its unit normal vector is continuous on each closed regular curve on the surface, [8].

The inner product of two vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$ is denoted by $\langle \mathbf{u}, \mathbf{v}\rangle$. Similarly, the plane through a point $\mathbf{p}$ in $\mathbb{R}^3$ spanned by two linearly-independent vectors $\mathbf{u}, \mathbf{v}$ is denoted by $[\mathbf{p}, \mathbf{u}, \mathbf{v}]$.

For linearly-independent unit vectors $\mathbf{u}, \mathbf{v}$ and a unit vector $\mathbf{n}$ such that $\mathbf{n} \perp \mathbf{u}$ and $\mathbf{n} \perp \mathbf{v}$, we denote by $(\mathbf{u}, \mathbf{v})_{\mathbf{n}}$ the oriented angle between $\mathbf{u}$ and $\mathbf{v}$ in the sense of $\mathbf{n}$. Precisely, the angle $A = (\mathbf{u}, \mathbf{v})_{\mathbf{n}}$ is defined (see Fig. 1) by

$$\sin A = \det(\mathbf{u}, \mathbf{v}, \mathbf{n}), \qquad \cos A = \langle \mathbf{u}, \mathbf{v}\rangle . \tag{1.1}$$

The variable s is employed to denote arc length along a space curve. Note that the arc-length parameterization $\mathbf{r} : s \to \mathbf{r}(s)$ of a curve satisfies $\|\mathbf{r}'(s)\| = 1$ and $\mathbf{r}'(s) \perp \mathbf{r}''(s)$ for all s. However, in this paper, a general parameterization $\mathbf{r} : t \to \mathbf{r}(t)$ is often used in the surface construction problem. The parameters of functions may sometimes be omitted when no confusion can arise.

· With each point $\mathbf{r}(s)$ of a curve satisfying $\mathbf{r}''(s) \neq 0$, we associate the *Serret–Frenet frame* $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{b}(s))$ where $\mathbf{T}(s) = \mathbf{r}'(s), \mathbf{N}(s) = \mathbf{r}''(s)/\|\mathbf{r}''(s)\|$, and $\mathbf{b}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ are, respectively, the unit *tangent, principal normal*, and

2000 *Mathematics Subject Classification:* 53A04, 53A10 *binormal* vectors of the curve at the point $\mathbf{r}(s)$. The arc-length derivative of the Serret–Frenet frame is governed by the relations

$$\frac{d}{ds}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{b}(s)\end{bmatrix}=\begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{b}(s)\end{bmatrix}, \tag{1.2}$$

where the *curvature* $\kappa(s)$ and *torsion* $\tau(s)$ of the curve $\mathbf{r}(s)$ are defined by

$$\kappa(s)=\|\mathbf{r}''(s)\| \quad \text{and} \quad \tau(s)=\frac{\det(\mathbf{r}'(s),\mathbf{r}''(s),\mathbf{r}'''(s))}{\|\mathbf{r}''(s)\|^2}. \tag{1.3}$$

The osculating plane at each curve point $\mathbf{r}(s)$ is spanned by the two vectors $\mathbf{T}(s),\mathbf{N}(s)$ and does not depend on the curve parameterization. If $\kappa(s)=0$ for some s, then $\mathbf{r}''(s)=0$ and the normal vector $\mathbf{n}(s)$ and osculating plane are undefined at that point. This condition identifies an *inflection* of the curve, [8].

· On a regular oriented surface $(u,v)\to\mathbf{R}(u,v)$, the unit normal is defined at each point in terms of the partial derivatives $\mathbf{R}_u=\partial\mathbf{R}/\partial u,\mathbf{R}_v=\partial\mathbf{R}/\partial v$ by

$$\mathbf{n}(u,v)=\frac{\mathbf{R}_u(u,v)\times\mathbf{R}_v(u,v)}{\|\mathbf{R}_u(u,v)\times\mathbf{R}_v(u,v)\|}. \tag{1.4}$$

· Consider a curve $\mathbf{r}(s)=\mathbf{R}((u(s),v(s))$ on a surface $\mathbf{R}(u,v)$, where s denotes arc length for the space curve $\mathbf{r}(s)$, but not necessarily for the plane curve defined by $s\to((u(s),v(s))$. With each point $\mathbf{r}(s)$ we associate the *Darboux frame* $(\mathbf{T}(s),\mathbf{P}(s),\mathbf{n}(s))-$ where $\mathbf{T}(s)$ is the unit tangent vector of the curve. $\mathbf{n}(s)$ is the unit normal vector of the surface at the point $\mathbf{R}((u(s),v(s))=\mathbf{r}(s)$, and $\mathbf{P}(s)=\mathbf{n}(s)\times\mathbf{T}(s)$. The arc-length derivative of the Darboux frame is given by the relations

$$\frac{d}{ds}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{P}(s)\\ \mathbf{n}(s)\end{bmatrix}=\begin{bmatrix}0 & \kappa_g(s) & \kappa_n(s)\\ -\kappa_g(s) & 0 & \tau_g(s)\\ -\kappa_n(s) & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{P}(s)\\ \mathbf{n}(s)\end{bmatrix}, \tag{1.5}$$

which define the *normal curvature* $\kappa_n(s)$, the *geodesic curvature* $\kappa_g(s)$, and the *geodesic torsion* $\tau_g(s)$ at each point of the curve $\mathbf{r}(s)$ as

$$\kappa_n=\left\langle\frac{d\mathbf{T}}{ds},\mathbf{n}\right\rangle,\quad \kappa_g=\left\langle\frac{d\mathbf{T}}{ds},\mathbf{P}\right\rangle,\quad \tau_g=\left\langle\frac{d\mathbf{T}}{ds},\mathbf{n}\right\rangle. \tag{1.6}$$

A regular curve $t\to\mathbf{r}(t)$ *is a geodesic on the surface* $\mathbf{R}(u,v)$ *if and only if*

$(D1)$ the geodesic curvature of $\mathbf{r}(t)$ is identically zero;

$(D2)$ *the principal normal at each non-inflection point of* $\mathbf{r}(t)$ *is orthogonal to the surface tangent plane at the point* $\mathbf{R}((u(t),v(t))=\mathbf{r}(t)$;

$(D3)$ *the osculating plane at each non-inflection point of* $\mathbf{r}(t)$ *is orthogonal to the surface tangent plane at the point* $\mathbf{R}((u(t),v(t))=\mathbf{r}(t)$.

2. $\mathcal{D}-$Focal Curves According To Darboux Frame In $\mathbb{M}^3$

Denoting the focal curve by $\mathfrak{D}_\gamma$, we can write

$$\mathfrak{D}_\gamma(s) = (\gamma + \mathfrak{f}_1^{\mathcal{D}}\mathbf{P} + \mathfrak{f}_2^{\mathcal{D}}\mathbf{n})(s), \tag{2.1}$$

where the coefficients $\mathfrak{f}_1^{\mathcal{D}}$, $\mathfrak{f}_2^{\mathcal{D}}$ are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively.

To separate a focal curve according to Darboux frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as $\mathcal{D}$-focal curve.

Theorem 2.1. *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}^3$. Then,*

$$\begin{aligned} \mathfrak{D}_\gamma^{\mathcal{D}}(s) &= \gamma(s) + e^{-\int \frac{\tau_g\kappa_g}{\kappa_n}ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n}e^{\int \frac{\tau_g\kappa_g}{\kappa_n}ds}ds]\mathbf{P} \\ &\quad +[\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}e^{-\int \frac{\tau_g\kappa_g}{\kappa_n}ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n}e^{\int \frac{\tau_g\kappa_g}{\kappa_n}ds}ds]]\mathbf{n}, \end{aligned} \tag{2.2}$$

where $\mathfrak{C}$ is a constant of integration.

Proof: Assume that γ is a unit speed curve and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}^3$.

By differentiating of the formula (2.1), we get

$$\mathfrak{D}_\gamma^{\mathcal{D}}(s)' = (1 - \mathfrak{f}_1^{\mathcal{D}}\kappa_g - \mathfrak{f}_2^{\mathcal{D}}\kappa_n)\mathbf{T} + ((\mathfrak{f}_1^{\mathcal{D}})' - \mathfrak{f}_2^{\mathcal{D}}\tau_g)\mathbf{P} + (\mathfrak{f}_1^{\mathcal{D}}\tau_g + (\mathfrak{f}_2^{\mathcal{D}})')\mathbf{n}, \tag{2.3}$$

where the coefficients $\mathfrak{f}_1^{\mathcal{D}}$, $\mathfrak{f}_2^{\mathcal{D}}$ are smooth functions of the parameter of the curve γ.

Using above equation, the first 2 components vanish, we get

$$\begin{aligned} \mathfrak{f}_1^{\mathcal{D}}\kappa_g + \mathfrak{f}_2^{\mathcal{D}}\kappa_n &= 1, \\ (\mathfrak{f}_1^{\mathcal{D}})' - \mathfrak{f}_2^{\mathcal{D}}\tau_g &= 0. \end{aligned}$$

Considering second equation above system, we have

$$\mathfrak{f}_1^{\mathcal{D}} = \frac{1 - \mathfrak{f}_2^{\mathcal{D}}\kappa_n}{\kappa_g} \text{ and } \mathfrak{f}_2^{\mathcal{D}} = \frac{1 - \mathfrak{f}_1^{\mathcal{D}}\kappa_g}{\kappa_n}$$

Since, we immediately arrive at

$$\begin{aligned} (\mathfrak{f}_1^{\mathcal{D}})' - \tau_g\left(\frac{1 - \mathfrak{f}_1^{\mathcal{D}}\kappa_g}{\kappa_n}\right) &= 0, \\ (\mathfrak{f}_1^{\mathcal{D}})' + \mathfrak{f}_1^{\mathcal{D}}\left(\frac{\tau_g\kappa_g}{\kappa_n}\right) &= \frac{\tau_g}{\kappa_n}. \end{aligned}$$

By means of obtained equations, we express (2.2). This completes the proof. □

Corollary 2.2. *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}^3$. Then, the focal curvatures of $\mathfrak{F}_\gamma$ are*

$$\mathfrak{f}_1^{\mathcal{D}} = e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds],$$

$$\mathfrak{f}_2^{\mathcal{D}} = [\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n} e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]],$$

where $\mathfrak{C}$ is a constant of integration.

In the light of Theorem 2.1, we express the following corollary without proof:

Lemma 2.3. *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}^3$. If κ_n and κ_g are constant then, the focal curvatures of $\mathfrak{F}_\gamma$ are*

$$\mathfrak{f}_1^{\mathcal{D}} = -\frac{1}{\kappa_g} + \mathfrak{Q} e^{\frac{\tau_g \kappa_g}{\kappa_n} s}$$

$$\mathfrak{f}_2^{\mathcal{D}} = \frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} + \mathfrak{Q} e^{\frac{\tau_g \kappa_g}{\kappa_n} s}],$$

where $\mathfrak{Q}$ is a constant of integration.

Theorem 2.4. *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}^3$. If κ_n and κ_g are constant then,*

$$\mathfrak{D}_\gamma^{\mathcal{D}}(s) = \gamma(s) + [-\frac{1}{\kappa_g} + Q e^{\frac{\tau_g \kappa_g}{\kappa_n} s}]\mathbf{P} + [\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} + Q e^{\frac{\tau_g \kappa_g}{\kappa_n} s}]]\mathbf{n},$$

where $\mathfrak{Q}$ is a constant of integration.

Corollary 2.5. *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}^3$. If γ is a principal line then,*

$$\mathfrak{D}_\gamma^{\mathcal{D}}(s) = \gamma(s) + \mathfrak{A}\mathbf{P} + [\frac{1 - \mathfrak{A}\kappa_g}{\kappa_n}]\mathbf{n},$$

where $\mathfrak{A}$ is a constant of integration.

References

1. P. Alegre , K. Arslan, A. Carriazo , C. Murathan and G. Öztürk: *Some Special Types of Developable Ruled Surface,* Hacettepe Journal of Mathematics and Statistics, 39 (3) (2010), 319 – 325.

2. A. A. Ergin: *Timelike Darboux curves on a timelike surface* $\mathbb{M} \subset \mathbb{M}_1^3$, Hadronic Journal 24(6) (2001) 701–712.

3. S. Bas and T. Körpınar: *Inextensible Flows of Spacelike Curves on Spacelike Surfaces according to Darboux Frame in* $\mathbb{M}_1^3$, Bol. Soc. Paran. Mat. 31 (2) (2013), 9–17.

4. Ö. Bektaş, M., S. Yüce: *Smarandache Curves According to Darboux Frame in Euclidean 3-Space.* arXiv: 1203.4830v1 [math. DG], 20 March 2012.

5. D. E. Blair: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag 509, Berlin-New York, 1976.

6. J.P.Cleave: *The form of the tangent developable at points of zero torsion on space curves,* Math. Proc. Camb. Phil. 88 (1980), 403–407.

7. N. Ekmekci and K. Ilarslan: *Null general helices and submanifolds,* Bol. Soc. Mat. Mexicana 9 (2) (2003), 279-286.

8. R.T. Farouki, N. Szafran, L. Biard: *Existence conditions for Coons patches interpolating geodesic boundary curves.* Computer-Aided Design, 26 (2009), 599–614.

9. O. Gursoy: *Some results on closed ruled surfaces and closed space curves,* Mech. Mach. Theory 27 (1990), 323–330.

10. D. J. Struik: *Lectures on Classical Differential Geometry,* Dover, New-York, 1988.

11. T. Körpınar and E. Turhan: *On Spacelike Biharmonic Slant Helices According to Bishop Frame in the Lorentzian Group of Rigid Motions* $\mathbb{E}(1,1)$, Bol. Soc. Paran. Mat. 30 (2) (2012), 91–100.

12. L. Kula and Y. Yayli: *On slant helix and its spherical indicatrix,* Applied Mathematics and Computation. 169 (2005), 600-607.

13. M. Khalifa Saad, H. S. Abdel-Aziz, G. Weiss, M. Solimman: *Relations among Darboux Frames of Null Bertrand Curves in Pseudo-Euclidean Space.* 1st Int. WLGK11, April 25-30, Paphos, Cyprus, 2011.

14. M. A. Lancret: *Memoire sur les courbes 'a double courbure,* Memoires presentes alInstitut 1 (1806), 416-454.

15. E. Loubeau and S. Montaldo: *Biminimal immersions in space forms,* preprint, 2004, math.DG/0405320 v1.

16. A. W. Nutbourne and R. R. Martin, *Differential Geometry Applied to the Design of Curves and Surfaces*, Ellis Horwood, Chichester, UK, 1988.

17. Y. Ou and Z. Wang: *Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces,* Mediterr. j. math. 5 (2008), 379–394.

18. N. Rahmani and S. Rahmani: *Lorentzian Geometry of the Heisenberg Group*, Geometriae Dedicata 118 (1) (2006) 133–140.

19. E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space* $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 99–104.

20. R. Uribe-Vargas: *On vertices, focal curvatures and differential geometry of space curves,* Bull. Brazilian Math. Soc. 36 (3) (2005), 285–307.

Vedat Asil, Selcuk Bas and Talat Körpinar
Fırat University, Department of Mathematics
23119, Elazığ, Turkey
E-mail address: selcukbas79@gmail.com