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$\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ Curves of Biharmonic New Type $\mathfrak{b}-$Slant Helices according to Bishop Frame in the Sol Space $\boldsymbol{Sol}^3$

Talat Körpınar and Essin Turhan

ABSTRACT: In this paper, we study $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. We characterize the $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the $\boldsymbol{Sol}^3$.

Key Words: new type $\mathfrak{b}-$slant helix, Sol space, curvatures

Contents

1. Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu.

This study is organised as follows: Firstly, we study $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. Secondly, we characterize the $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves in terms of their Bishop curvatures. Finally, we find explicit equations of $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves in the $\boldsymbol{Sol}^3$.

2. Riemannian Structure of Sol Space $\boldsymbol{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\boldsymbol{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ [11,12].

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Note that the Sol metric can also be written as:

$$g_{\boldsymbol{Sol}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i,$$

where

$$\boldsymbol{\omega}^1 = e^z dx, \quad \boldsymbol{\omega}^2 = e^{-z} dy, \quad \boldsymbol{\omega}^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\boldsymbol{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\boldsymbol{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x,y,z) &\rightarrow (x+c, y, z), \\ (x,y,z) &\rightarrow (x, y+c, z), \\ (x,y,z) &\rightarrow \left(e^{-c}x, e^{c}y, z+c\right). \end{aligned}$$

3. Biharmonic New Type $\mathfrak{b}-$Slant Helices in Sol Space $\boldsymbol{Sol}^3$

Assume that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion [14,15] and

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1, \; g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{n}) = 1, \; g_{\boldsymbol{Sol}^3}(\mathbf{b},\mathbf{b}) = 1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{n}) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{b}) = g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{b}) = 0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, [1]. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 + k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_1) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_2,\mathbf{m}_2) = 1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_1) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_2) = g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\delta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \delta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{3.5}$$

Theorem 3.1. *$\gamma : I \longrightarrow \boldsymbol{Sol}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \textit{constant} \neq 0, \\ k_1'' - \left[k_1^2 + k_2^2\right]k_1 &= -k_1\left[2m_2^3 - 1\right] - 2k_2m_1^3m_2^3, \\ k_2'' - \left[k_1^2 + k_2^2\right]k_2 &= 2k_1m_1^3m_2^3 - k_2\left[2m_1^3 - 1\right]. \end{aligned} \tag{3.6}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that new type slant helices have unit speed. The new type slant helices can be identified by a simple condition on natural curvatures.

To separate a new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as new type $\mathfrak{b}-$slant helix.

Theorem 3.2. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix with constant slope. Then, the position vector of γ is*

$$\begin{aligned} \gamma(s) = {} & [\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\mathcal{S}_1\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]]+\mathcal{S}_4 e^{-\sin\mathcal{M}s+\mathcal{S}_3}]\mathbf{e}_1 \\ & +[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\mathcal{S}_1\sin[\mathcal{S}_1 s+\mathcal{S}_2]]+\mathcal{S}_5 e^{\sin\mathcal{M}s-\mathcal{S}_3}]\mathbf{e}_2 \\ & +[-\sin\mathcal{M}s + \mathcal{S}_3]\mathbf{e}_3, \end{aligned} \tag{3.7}$$

where $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5$ are constants of integration, [8].

We can use Mathematica in Theorem 3.2, yields

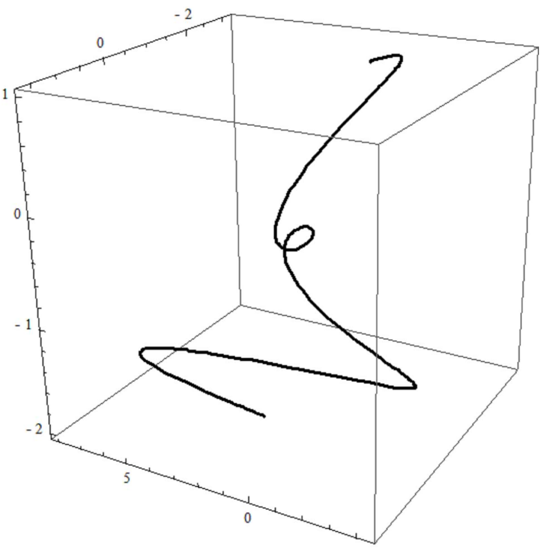

Fig. 1

4. $\mathcal{B}-$Smarandache $\mathbf{tm}_2$ Curves of Biharmonic New Type $\mathfrak{b}-$Slant Helices in $\boldsymbol{Sol}^3$

To separate a Smarandache $\mathbf{tm}_2$ curve according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curve.

Definition 4.1. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix and $\{\mathbf{t}, \mathbf{m_1}, \mathbf{m_2}\}$ be its moving Bishop frame. $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves are defined by*

$$\gamma_{\mathbf{tm}_2} = \frac{1}{\sqrt{k_1^2 + 2k_2^2}} \left(\mathbf{t} + \mathbf{m}_2\right). \tag{4.1}$$

Theorem 4.2. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the equation of $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves of biharmonic new type $\mathfrak{b}-$slant helix is given by*

$$\begin{aligned} \gamma_{\mathbf{tm}_2}(s) \quad = \quad & \frac{1}{\sqrt{k_1^2 + 2k_2^2}} [\cos \mathcal{M} \sin [\mathcal{S}_1 s + \mathcal{S}_2] + \sin \mathcal{M} \sin [\mathcal{S}_1 s + \mathcal{S}_2]]\mathbf{e}_1 \\ & + \frac{1}{\sqrt{k_1^2 + 2k_2^2}} [\cos \mathcal{M} \cos [\mathcal{S}_1 s + \mathcal{S}_2] + \sin \mathcal{M} \cos [\mathcal{S}_1 s + \mathcal{S}_2]]\mathbf{e}_2 \quad (4.2) \\ & + \frac{1}{\sqrt{k_1^2 + 2k_2^2}} [\cos \mathcal{M} - \sin \mathcal{M}]\mathbf{e}_3, \end{aligned}$$

where $\mathcal{C}_1, \mathcal{C}_2$ are constants of integration.

Proof: Assume that γ is a non geodesic biharmonic new type $\mathfrak{b}-$slant helix according to Bishop frame.

From Theorem 3.2, we obtain

$$\mathbf{m}_2 = \sin\mathcal{M}\sin\left[\mathcal{S}_1 s + \mathcal{S}_2\right]\mathbf{e}_1 + \sin\mathcal{M}\cos\left[\mathcal{S}_1 s + \mathcal{S}_2\right]\mathbf{e}_2 + \cos\mathcal{M}\mathbf{e}_3, \tag{4.3}$$

where $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{R}$.

Using Bishop frame, we have

$$\mathbf{t} = \cos\mathcal{M}\sin\left[\mathcal{S}_1 s + \mathcal{S}_2\right]\mathbf{e}_1 + \cos\mathcal{M}\cos\left[\mathcal{S}_1 s + \mathcal{S}_2\right]\mathbf{e}_2 - \sin\mathcal{M}\mathbf{e}_3. \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.1) we have (4.2), which completes the proof. $\square$

In terms of Eqs. (2.1) and (4.2), we may give:

Corollary 4.3. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the parametric equations of $\mathfrak{b}-$Smarandache $\mathbf{tm}_2$ curves of biharmonic new type $\mathfrak{b}-$slant helix are given by*

$$\begin{aligned}
x_{\mathbf{tm}_2}(s) &= \frac{e^{-\frac{1}{\sqrt{k_1^2+2k_2^2}}[\cos\mathcal{M}-\sin\mathcal{M}]}}{\sqrt{k_1^2+2k_2^2}}[\cos\mathcal{M}\sin\left[\mathcal{S}_1 s + \mathcal{S}_2\right] + \sin\mathcal{M}\sin\left[\mathcal{S}_1 s + \mathcal{S}_2\right]], \\
y_{\mathbf{tm}_2}(s) &= \frac{e^{\frac{1}{\sqrt{k_1^2+2k_2^2}}[\cos\mathcal{M}-\sin\mathcal{M}]}}{\sqrt{k_1^2+2k_2^2}}[\cos\mathcal{M}\cos\left[\mathcal{S}_1 s + \mathcal{S}_2\right] + \sin\mathcal{M}\cos\left[\mathcal{S}_1 s + \mathcal{S}_2\right]], \quad (4.5) \\
z_{\mathbf{tm}_2}(s) &= \frac{1}{\sqrt{k_1^2+2k_2^2}}[\cos\mathcal{M}-\sin\mathcal{M}],
\end{aligned}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are constants of integration.

Proof: Substituting (2.1) to (4.2), we have (4.5) as desired. $\square$

We may use Mathematica in Corollary 4.3, yields

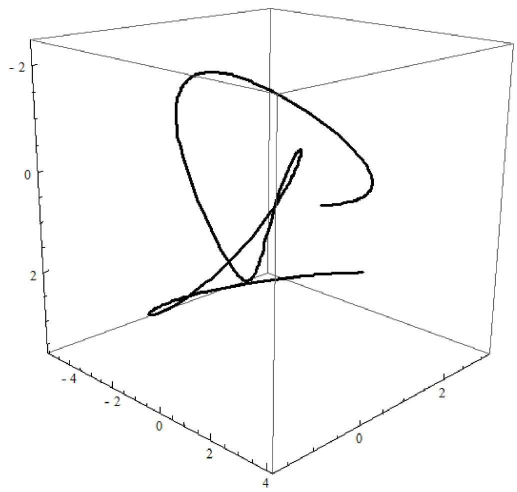

Fig. 2

References

1. L. R. Bishop: *There is More Than One Way to Frame a Curve,* Amer. Math. Monthly 82 (3) (1975) 246-251.

2. R.M.C. Bodduluri, B. Ravani: *Geometric design and fabrication of developable surfaces,* ASME Adv. Design Autom. 2 (1992) , 243-250.

3. F. Dillen, W. Kuhnel: *Ruled Weingarten surfaces in Minkowski 3-space,* Manuscripta Math. 98 (1999), 307–320.

4. I. Dimitric: *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.

5. J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1–68.

6. J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

7. G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

8. T. Körpınar and E. Turhan: *Biharmonic new type b-slant helices according to Bishop frame in the sol space, (submitted).*

9. M. A. Lancret: *Memoire sur les courbes ‘a double courbure,* Memoires presentes alInstitut 1 (1806), 416-454.

10. E. Loubeau and S. Montaldo: *Biminimal immersions in space forms,* preprint, 2004, math.DG/0405320 v1.

11. Y. Ou and Z. Wang: *Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces,* Mediterr. j. math. 5 (2008), 379–394

12. S. Rahmani: *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.

13. D. J. Struik: *Lectures on Classical Differential Geometry,* Dover, New-York, 1988.

14. E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis*^{3}, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

15. E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space* $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 99–104.

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