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$C-$sets and decomposition of continuity in generalized topological spaces

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abstract: In generalized topological spaces, we define $\mu C-$sets and establish some decomposition of continuity between generalized topological spaces. Moreover, we prove that some of the results established in [3] are already established results in topological spaces [10]. Some generalizations of these results are also given.

Key Words: generalized topology; $\mu-$open and $\mu-$closed sets; quasi-topology; $\mu\alpha-$open, $\mu\sigma-$open, $\mu\pi-$open, $\mu b-$open and $\mu\beta-$open sets.

Contents

1. Introduction

A nonempty family μ of subsets of a set X is said to be a *generalized topology* [5] if $\emptyset \in \mu$ and arbitrary union of elements of μ is again in μ. The pair (X,μ) is called a generalized topological space or simply, a space and elements of μ are called $\mu-$*open* sets. $A \subset X$ is a $\mu-$*closed set* if $X - A$ is a $\mu-$open set. If $X \in \mu$, then (X,μ) is called a *strong* [7] space. In a space (X,μ), if μ is closed under finite intersection, then (X,μ) is called a *quasi-topological space* [9]. Clearly, every strong, quasi-topological space is a topological space. For $A \subset X$, $c_\mu(A)$ is the intersection of all $\mu-$closed sets containing A and $i_\mu(A)$ is the union of all $\mu-$open sets contained in A. Moreover, $X - c_\mu(A) = i_\mu(X - A)$, for every subset A of X. Clearly, (X,μ) is strong if and only if $\emptyset$ is $\mu-$closed if and only if $c_\mu(\{\emptyset\}) = \emptyset$.

A subset A of a space (X,μ) is said to be a $\mu\alpha-$open set [8](resp. $\mu\sigma-$open set [8], $\mu\pi-$open set [8], $\mu b-$open set [15], $\mu\beta-$open set [8]) if $A \subset i_\mu c_\mu i_\mu(A)$ (resp. $A \subset c_\mu i_\mu(A)$, $A \subset i_\mu c_\mu(A)$, $A \subset i_\mu c_\mu(A) \cup c_\mu i_\mu(A)$, $A \subset c_\mu i_\mu c_\mu(A)$). A subset A of a space(X,μ) is said to be a $\mu\alpha-$closed set (resp. $\mu\sigma-$closed set, $\mu\pi-$closed set, $\mu b-$closed set, $\mu\beta-$closed set) if $X - A$ is a $\mu\alpha-$open set (resp. $\mu\sigma-$open set, $\mu\pi-$open set, $\mu b-$open set, $\mu\beta-$open set). We will denote the family of all $\mu\alpha-$open sets (resp. $\mu\sigma-$open sets, $\mu\pi-$open sets, $\mu b-$open sets, $\mu\beta-$open sets)

2000 *Mathematics Subject Classification:* 54A05, 54C08 by $\alpha(\mu)$ (resp. $\sigma(\mu)$, $\pi(\mu)$, $b(\mu)$, $\beta(\mu)$).

A subset A of a topological space (X,τ) is said to be $\alpha-$open [14](resp. semiopen [12], preopen [13], $b-$open [1], $\beta-$open [2]) if $A \subset ici(A)$ (resp. $A \subset ci(A)$, $A \subset ic(A)$, $A \subset ic(A) \cup ci(A)$, $A \subset cic(A)$) where c and i are respectively, the closure and interior operators of the topological space (X,τ).

If μ is a monotonic function defined on $\wp(X)$ of a nonempty set X, then a subset A of X is said to be *$\mu-$regular* [4] if $A = \mu(A)$. More generally, A is said to be *$\mu\pi-$regular* if $A = i_\mu c_\mu(A)$. A subset A of a space (X,μ) is said to be *$b_\mu-$locally closed set* if $A = U \cap S$ where U is a $\mu-$open set and S is a $\mu b-$closed set. A subset S of X is said to be a *$\mu t-$set* [11] if $i_\mu c_\mu(S) = i_\mu(S)$. A subset S of X is said to be a *$\mu B-$set* [11] if there is a $\mu-$open set U and a $\mu t-$set A such that $S = U \cap A$. The family of all $\mu B-$ sets in a space is denoted by $B(X,\mu)$. The following lemmas will be useful in the sequel.

Lemma 1.1. *If (X,μ) is a quasi-topological space, then the following hold.*
(a) If A and B are $\mu-$open sets, then $A \cap B$ is a $\mu-$open set [15, Theorem 2.1].
(b) $i_\mu(A \cap B) = i_\mu(A) \cap i_\mu(B)$ for every subsets A and B of X [9, Theorem 2.2].
(c) $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$ for every subsets A and B of X [15, Theorem 2.3(b)].

Lemma 1.2. *[11] Let (X,μ) be a quasi-topological space and $A \subset X$. A is a $\mu-$open set if and only if it is both a $\mu\pi-$open set and a $\mu B-$set.*

Lemma 1.3. *[11] Let (X,μ) be a space and $A \subset X$. A is a $\mu B-$set if and only if there exists a $\mu-$open set U such that $H = U \cap c_\sigma(H)$.*

2. $\mu C-$sets

Let (X,μ) be a space and $S \subset X$. S is said to be a $\mu t^\star-$set if $i_\mu c_\mu i_\mu(S) = i_\mu(S)$. The family of all $\mu t^\star-$sets of a space (X,μ) is denoted by $t^\star(X,\mu)$. The following Theorem 2.1 gives characterizations of $\mu t^\star-$sets. If $M_\mu = \cup\{M \mid M \in \mu\}$, then M_μ is the largest $\mu-$open set contained in X. Since $i_\mu c_\mu i_\mu(X) = i_\mu(X)$, X is a $\mu t^\star-$set.

Theorem 2.1. *Let (X,μ) be a space and $S \subset X$. Then the following are equivalent.*
(a) $S \in t^\star(X,\mu)$.
(b) S is a $\mu\beta-$closed set.
(c) $i_\mu(S)$ is a $\mu\pi-$regular set.

Proof: (a)⇒(b). Let $S \in t^\star(X,\mu)$. Then $i_\mu c_\mu i_\mu(S) = i_\mu(S) \subset S$ and so S is a $\mu\beta-$closed set.
(b)⇒(c). Suppose that S is a $\mu\beta-$closed set. Then $i_\mu c_\mu i_\mu(S) \subset S$ which implies that $i_\mu c_\mu i_\mu(S) \subset i_\mu(S)$. Clearly, $i_\mu(S) \subset i_\mu c_\mu i_\mu(S)$. Therefore, $i_\mu c_\mu i_\mu(S) = i_\mu(S)$ which implies that $i_\mu(S)$ is a $\mu\pi-$regular set.
(c)⇒(a). Since $i_\mu(S)$ is a $\mu\pi-$regular set, $i_\mu(S) = i_\mu c_\mu i_\mu(S)$. Therefore, $S \in t^\star(X,\mu)$. □

The following Theorem 2.2 gives some properties of $\mu t^\star-$sets.

Theorem 2.2. *Let* (X,μ) *be a space and* $S\subset X$. *Then the following hold.*
(a) If S *is a* $\mu t-$*set, then* S *is a* $\mu t^{\star}-$*set.*
(b) If S *is a* $\mu\sigma-$*open set, then* S *is a* $\mu t-$*set if and only if* S *is a* $\mu t^{\star}-$*set.*
(c) $S\in\alpha\cap t^{\star}(X,\mu)$ *if and only if* S *is a* $\mu\pi-$*regular set.*

Proof: (a) If S is a $\mu t-$set, then $i_\mu(S)=i_\mu c_\mu(S)$ and so $i_\mu c_\mu i_\mu(S)=i_\mu c_\mu i_\mu c_\mu(S)=i_\mu c_\mu(S)=i_\mu(S)$. Therefore, S is a $\mu t^{\star}-$set.
(b) If S is a $\mu t^{\star}-$set, then $i_\mu(S)=i_\mu c_\mu i_\mu(S)$. Since S is a $\mu\sigma-$open set, $c_\mu i_\mu(S)=c_\mu(S)$ and so $i_\mu c_\mu(S)=i_\mu c_\mu i_\mu(S)=i_\mu(S)$. Therefore, S is a $\mu t-$set. Converse follows from (a).
(c) If S is a $\mu t^{\star}-$set, then $i_\mu c_\mu i_\mu(S)=i_\mu(S)\subset S$. Since S is a $\mu\alpha-$open set, $S=i_\mu c_\mu i_\mu(S)$ and so $i_\mu c_\mu(S)=i_\mu c_\mu i_\mu(S)=S$. Therefore, S is a $\mu\pi-$regular set. Conversely, suppose S is a $\mu\pi-$regular set. Then S is a $\mu-$open set and so S is a $\mu\alpha-$open set. Since $i_\mu c_\mu i_\mu(S)=i_\mu c_\mu(S)=S=i_\mu(S)$, it follows that $S\in\alpha\cap t^{\star}(X,\mu)$. □

The following Example 2.3 shows that the converse of 2.2(a) is false. Example 2.4 below shows that the concepts $\mu\alpha-$open set and $\mu t^{\star}-$set are independent. Also, it shows that a $\mu-$open set need not be a $\mu t^{\star}-$set. Example 2.5 below shows that the union of two $\mu t^{\star}-$sets need not be a $\mu t^{\star}-$set.

Example 2.3. *Let* $X=\{a,b,c,d\}$ *and* $\mu=\{\emptyset,\{d\},\{b,c\},\{b,c,d\}\}$. *The family of all* $\mu-$*closed sets are* $\{X,\{a,b,c\},\{a,d\},\{a\}\}$. *Let* $A=\{a,b\}$. *Then* $i_\mu(A)=\emptyset$ *and* $i_\mu c_\mu i_\mu(A)=i_\mu c_\mu(\{\emptyset\})=i_\mu(\{a\})=\emptyset$. *Therefore,* $i_\mu(A)=i_\mu c_\mu i_\mu(A)$ *and so* A *is a* $\mu t^{\star}-$*set. Now* $i_\mu c_\mu(A)=i_\mu(\{a,b,c\})=\{b,c\}\neq i_\mu(A)$. *Therefore,* A *is not a* $\mu t-$*set.*

Example 2.4. *Let* $X=\{a,b,c,d\}$ *and* $\mu=\{\emptyset,\{c\},\{d\},\{c,d\},\{a,c,d\}\}$. *The family of all* $\mu-$*closed sets are* $\{X,\{a,b,d\},\{a,b,c\},\{a,b\},\{b\}\}$. *Let* $A=\{c,d\}$. *Then* $i_\mu c_\mu i_\mu(A)=i_\mu c_\mu(\{c,d\})=i_\mu(X)=\{a,c,d\}$. *Therefore,* A *is a* $\mu\alpha-$*open set. But* $i_\mu c_\mu i_\mu(A)\neq i_\mu(A)$. *Therefore,* A *is not a* $\mu t^{\star}-$*set. Let* $B=\{a,b\}$. *Then* $i_\mu c_\mu i_\mu(B)=i_\mu c_\mu(\{\emptyset\})=i_\mu(\{b\})=i_\mu(B)$. *Therefore,* B *is a* $\mu t^{\star}-$*set. But* B *is not a* $\mu\alpha-$*open set. This also shows that a* $\mu-$*open set need not be a* $\mu t^{\star}-$*set.*

Example 2.5. *Consider the space in Example 2.4. If* $A=\{c\}$ *and* $B=\{d\}$, *then* $A\cup B=\{c,d\}$, $i_\mu(A)=A$, $i_\mu(B)=B$. *Here,* $i_\mu c_\mu i_\mu(A)=i_\mu c_\mu(\{c\})=i_\mu(\{a,b,c\})=\{c\}=A=i_\mu(A)$. *Therefore,* A *is a* $\mu t^{\star}-$*set. Now* $i_\mu c_\mu i_\mu(B)=i_\mu c_\mu(\{d\})=i_\mu(\{a,b,d\})=\{d\}=B=i_\mu(B)$. *Therefore,* B *is a* $\mu t^{\star}-$*set. Also,* $i_\mu c_\mu i_\mu(\{c,d\})=i_\mu c_\mu(\{c,d\})=i_\mu(X)=\{a,c,d\}\neq i_\mu(\{c,d\})$. *Therefore,* $A\cup B$ *is not a* $\mu t^{\star}-$*set.*

Let (X,μ) be a space and $S\subset X$. S is said to be a $\mu C-$set if there exist a $\mu-$open set U and $A\in t^{\star}(X,\mu)$ such that $S=U\cap A$. The family of all $\mu C-$ sets in a space (X,μ) is denoted by $C(X,\mu)$. Since X is a $\mu t^{\star}-$set, every $\mu-$open set is a $\mu C-$ set. If (X,μ) is strong, then every $\mu t^{\star}-$set is a $\mu C-$set. The following Example 2.6 shows that the condition strong on the space (X,μ) cannot be dropped. Example 2.7 below shows that a $\mu C-$ set need not be a $\mu t^{\star}-$set and Example 2.8 below shows that a $\mu C-$ set need not be a $\mu-$open set.

Example 2.6. *Consider the space in Example 2.4 which is not strong. If* $A = \{a, b\}$, *then* $i_\mu(A) = \emptyset$. $i_\mu c_\mu i_\mu(A) = i_\mu c_\mu(\emptyset) = i_\mu(\{b\}) = \emptyset = i_\mu(A)$. *Therefore,* $A \in t^\star(X, \mu)$. *But* $\{a, b\} \notin C(X, \mu)$, *since there is no open set containing* $\{a, b\}$.

Example 2.7. *Consider the space in Example 2.4. If* $A = \{c, d\}$, *then* $i_\mu(A) = A$ *and* $i_\mu c_\mu i_\mu(A) = i_\mu c_\mu(A) = i_\mu(X) = \{a, c, d\} \neq i_\mu(A)$. *Therefore,* A *is not a* $\mu t^\star-$*set. Also,* $\{c, d\} = \{c, d\} \cap \{a, c, d\}$ *where* $\{c, d\}$ *is a* $\mu-$*open set and* $i_\mu c_\mu i_\mu(\{a, c, d\}) = i_\mu c_\mu(\{a, c, d\}) = i_\mu(X) = \{a, c, d\} = i_\mu(\{a, c, d\})$. *Therefore,* $A = \{c, d\}$ *is a* $\mu C-$*set.*

Example 2.8. *Consider the space in Example 2.4.* $\{a\}$ *is not a* $\mu-$*open set. Here* $\{a\} = \{a\} \cap \{a, c, d\}$ *where* $\{a, c, d\}$*is a* $\mu-$*open set and since* $i_\mu c_\mu i_\mu(\{a\}) = i_\mu c_\mu(\{\emptyset\}) = i_\mu(\{b\}) = \emptyset = i_\mu(\{a\})$, $\{a\}$ *is a* $\mu t^\star-$*set. Therefore,* $\{a\}$ *is a* $\mu C-$*set.*

The proof of the following Lemma 2.9 is similar to Theorem 3.8 of [16].

Lemma 2.9. *[16, Theorem 3.8]Let* (X, μ) *be a quasi-topological space and* $A, B \subset X$. *If either* A *or* B *is a* $\mu\sigma-$*open set, then* $i_\mu c_\mu(A \cap B) = i_\mu c_\mu(A) \cap i_\mu c_\mu(B)$.

Let (X, μ) be a space. For $\kappa, \nu \in \{\mu,\ \alpha,\ \pi,\ \sigma,\ b,\ \beta\}$ and $\kappa \neq \nu$, define $D(\kappa, \nu) = \{B \subset X \mid i_\kappa(B) = i_\nu(B)\}$. The following Theorem 2.10 gives a property of $\mu C-$sets. Example 2.11 below shows that the inclusions in Theorem 2.10 are proper.

Theorem 2.10. *Let* (X, μ) *be a quasi-topological space. Then* $B(X, \mu) \subset C(X, \mu) \subset D(\mu, \alpha)$.

Proof: Since every $\mu t-$set is a $\mu t^\star-$set, $B(X, \mu) \subset C(X, \mu)$. Let $S \in C(X, \mu)$. Then $S = U \cap A$ where U is a $\mu-$open set and A is a $\mu t^\star-$set . Now $i_\mu c_\mu i_\mu(S) = i_\mu c_\mu i_\mu(U \cap A) = i_\mu c_\mu(i_\mu(U) \cap i_\mu(A)) = i_\mu c_\mu i_\mu(U) \cap i_\mu c_\mu i_\mu(A)$, by Lemma 2.9 and so $i_\mu c_\mu i_\mu(S) = i_\mu c_\mu(U) \cap i_\mu c_\mu i_\mu(A) = i_\mu c_\mu(U) \cap i_\mu(A)$. Also, $i_\alpha(S) = S \cap i_\mu c_\mu i_\mu(S) = S \cap i_\mu c_\mu(U) \cap i_\mu(A) = U \cap A \cap i_\mu c_\mu(U) \cap i_\mu(A) = U \cap i_\mu(A) = i_\mu(S)$. Therefore, $S \in D(\mu, \alpha)$. □

Example 2.11. *Consider the space in Example 2.3. Now* $\{b\} = \{b, c\} \cap \{b\}$ *where* $\{b, c\}$ *is a* $\mu-$*open set and, since* $i_\mu c_\mu i_\mu(\{b\}) = i_\mu c_\mu(\emptyset) = i_\mu(\{a\}) = \emptyset = i_\mu(\{b\}), \{b\}$ *is a* $\mu t^\star-$*set. Therefore,* $\{b\}$ *is a* $\mu C-$*set. Again,* $\{b\}$ *is not a* $\mu t-$*set, since* $i_\mu c_\mu(\{b\}) = i_\mu(\{a, b, c\}) = \{b, c\} \neq \emptyset = i_\mu(\{b\})$. *If* $S = \{a, b\}$, *then* $i_\mu(S) = \emptyset$ *and so* $i_\mu c_\mu i_\mu(\{a, b\}) = i_\mu c_\mu(\emptyset) = i_\mu(\{a\}) = \emptyset$. *Also,* $i_\alpha(S) = S \cap i_\mu c_\mu i_\mu(S) = \emptyset = i_\mu(S)$ *and so* $S \in D(\mu, \alpha)$. *Since there is no open set containing* $\{a, b\}$, $\{a, b\}$ *is not a* $\mu C-$*set.*

The following Theorem 2.12 gives a decomposition of $\mu-$open sets in a quasi-topological space. Examples 2.13 shows that the concepts $\mu\alpha-$ *open set* and $\mu C-$ *set* are independent.

Theorem 2.12. *Let* (X, μ) *be a quasi-topological space and* $A \subset X$. *Then the following are equivalent.*
(a) S *is a* $\mu-$*open set.*
(b) S *is both a* $\mu\alpha-$*open set and a* $\mu C-$*set.*

Proof: (a)⇒(b). Suppose that S is a $\mu-$open set. Clearly, S is both a $\mu\alpha-$open set and a $\mu C-$set.
(b)⇒(a). If S is both a $\mu\alpha-$open set and a $\mu C-$set, then $S \subset i_\mu c_\mu i_\mu(S)$ and $S = U \cap A$, U is a $\mu-$open set and A is a $\mu t^\star-$set. Therefore, $S \subset i_\mu c_\mu i_\mu(U \cap A) = i_\mu c_\mu(U \cap i_\mu(A)) = i_\mu c_\mu(U) \cap i_\mu c_\mu i_\mu(A) = i_\mu c_\mu(U) \cap i_\mu(A)$. Since $S \subset U$, $S \subset U \cap i_\mu c_\mu(U) \cap i_\mu(A) = U \cap i_\mu(A) = i_\mu(U \cap A) = i_\mu(S)$. Hence S is a $\mu-$open set. □

Example 2.13. *(a) Consider the space in Example 2.3. If $A = \{b, d\}$, then $i_\mu c_\mu i_\mu(A) = i_\mu c_\mu(\{d\}) = i_\mu(\{a, d\}) = \{d\}$ and so A is not a $\mu\alpha-$open set. Since A is a $\mu t^\star-$set and $\{b, d\} = \{b, d\} \cap \{b, c, d\}$, where $\{b, c, d\}$ is a $\mu-$open set, we have $\{b, d\}$ is a $\mu C-$set.*
(b)Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{b, c, d\}, \{b\}\}$. The family of all $\mu-$closed sets are $\{X, \{a\}, \{a, c, d\}\}$. Here, $i_\mu c_\mu i_\mu\{b, c\} = \{b, c, d\}$ and so $\{b, c\}$ is a $\mu\alpha-$open set. Since $\{b, c, d\}$ is the only $\mu-$open set such that $\{b, c\} = \{b, c, d\} \cap \{b, c\}$ and $\{b, c\}$ is not a $\mu t^\star-$set, $\{b, c\}$ is not a $\mu C-$set.

3. $b_\mu-$locally closed sets, $b - t-$sets and $b - B-$sets

The following Theorem 3.1 gives a decomposition of $\mu-$open sets and Example 3.2 below shows that the concepts $\mu b-$open set and $D(\mu, b)-$set are independent. Theorem 3.3 below gives a characterization of $b_\mu-$locally closed set.

Theorem 3.1. *Let (X, μ) be a space and $A \subset X$. Then the following are equivalent.*
(a) A is a $\mu-$open set.
(b) A is both a $\mu b-$open set and a $D(\mu, b)-$set.

Proof: (a)⇒(b). Suppose that A is a $\mu-$open set. Then clearly (b) follows.
(b)⇒(a). Suppose that A is both a $\mu b-$open set and a $D(\mu, b)-$set. Then $i_\mu(A) = i_b(A) = A$. Therefore, A is a $\mu-$open set. □

Example 3.2. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$. The family of all $\mu-$closed sets are $\{X, \{a, b, d\}, \{a, b, c\}, \{a, b\}, \{a\}\}$. Here, $c_\mu i_\mu\{a, b, d\}$ $= \{a, b, d\}$ and $i_\mu c_\mu\{a, b, d\} = \{d\}$. $i_\mu c_\mu\{a, b, d\} \cup c_\mu i_\mu\{a, b, d\} = \{a, b, d\}$. Therefore, $\{a, b, d\}$ is a $\mu b-$open set. Since $i_\mu(\{a, b, d\}) = \{d\}$, $\{a, b, d\} \notin D(\mu, b)$. Thus $\{a, b, d\}$ is a $\mu b-$open set but $\{a, b, d\} \notin D(\mu, b)$.*
(b) If $A = \{a\}$, then A is not a $\mu b-$open set and so $i_b(A) = \emptyset = i_\mu(A)$ which implies that $A \in D(\mu, b)$.

Theorem 3.3. *Let (X, μ) be a space and $H \subset X$. Then H is a $b_\mu-$locally closed set if and only if there exists a $\mu-$open set U such that $H = U \cap c_b(H)$.*

Proof: Let H be a $b_\mu-$locally closed set. Then $H = U \cap F$ where U is a $\mu-$open set and F is a $\mu b-$closed set. Since $H \subset U$ and $H \subset F$, we have $H \subset c_b(H) \subset c_b(F) = F$ and $H \subset U \cap c_b(H) \subset U \cap c_b(F) = U \cap F = H$. Hence $H = U \cap c_b(H)$. Conversely, suppose $H = U \cap c_b(H)$ for some $\mu-$open set U. Since $c_b(H)$ is a $\mu b-$closed set, H is a $b_\mu-$locally closed set. □

Theorem 3.4. *Let (X, μ) be a space and $A \subset X$. Then the following hold.*
(a)A is a $b_\mu-$locally closed set.
(b) $c_b(A) - A$ is a $\mu b-$closed set.
(c) $A \cup (X - c_b(A))$ is a $\mu b-$open set.
(d) $A \subset i_b(A \cup (X - c_b(A)))$.

Proof: (a)⇒(b). Since A is a $b_\mu-$locally closed set, by Theorem 3.3, there exists a $\mu-$open set U such that $A = U \cap c_b(A)$. Now $c_b(A) - A = c_b(A) - (U \cap c_b(A)) = c_b(A) \cap (X - (U \cap c_b(A))) = c_b(A) \cap ((X - U) \cup (X - c_b(A)) = c_b(A) \cap (X - U)$ which is a $\mu b-$closed set.
(b)⇒(c). Since $c_b(A) - A$ is $\mu b-$closed, $X - (c_b(A) - A)$ is a $\mu b-$open set. Now $X-(c_b(A)-A) = X-((c_b(A)\cap(X-A)) = A\cup(X-c_b(A))$. Therefore, $A\cup(X-c_b(A))$ is a $\mu b-$open set.
(c)⇒(d). Since $A \subset A \cup (X - c_b(A))$, $A \subset i_b(A \cup (X - c_b(A)))$.
□

Let (X, μ) be a space. Subsets A and B of X are said to be $\mu-$*separated* [6] if $A\cap c_\mu(B) = \emptyset$ and $B\cap c_\mu(A) = \emptyset$. We know that $b(\mu)$ is a generalized topology. Since X is a $\mu\sigma-$open set, it follows that $X \in b(\mu)$ and so $(X, b(\mu))$ is a strong space. In addition, if $b(\mu)$ is closed under finite intersection, then $(X, b(\mu))$ is a topological space. The following Theorem uses this fact and discuss about the union of two $b_\mu-$locally closed sets.

Theorem 3.5. *Let (X, μ) be a quasi-topological space such that $(X, b(\mu))$ be a topological space and A and B be $b_\mu-$locally closed sets. If A and B are $\mu-$separated sets, then $A \cup B$ is a $b_\mu-$locally closed set.*

Proof: Since A and B are $b_\mu-$locally closed sets, $A = G\cap c_b(A)$ and $B = H\cap c_b(B)$ where G and H are $\mu-$open sets in X. Put $U = G\cap(X - c_\mu(B))$ and $V = H\cap(X - c_\mu(A))$. Then $U\cap c_b(A) = (G\cap(X-c_\mu(B)))\cap c_b(A) = (G\cap c_b(A))\cap(X-c_\mu(B)) = A\cap(X-c_\mu(B)) = A$, since $A \subset X - c_\mu(B)$. Moreover, $U\cap c_b(B) \subset U\cap c_\mu(B) = \emptyset$. Similarly, we can prove that $V \cap c_b(B) = B$ and $V \cap c_b(A) = \emptyset$. Since U and V are $\mu-$open sets, $(U\cup V)\cap c_b(A\cup B) = (U\cup V)\cap(c_b(A)\cup c_b(B)) = ((U\cap c_b(A))\cup(V\cap c_b(A)))\cup((V\cap c_b(B))\cup(U\cap c_b(B))) = A\cup B$. Therefore, $A\cup B$ is a $b_\mu-$locally closed set. □

The following Lemma 3.6 is essential to prove Theorem 3.7 below which is in turn used to prove Theorem 3.8 below which gives a decomposition of $\mu-$open sets. Example 3.9 below shows that the notions $\mu\alpha-$*open set* and $b_\mu-$*locally closed set* are independent.

Lemma 3.6. *Let (X, μ) be a quasi-topological space. Then the following hold.*
(a)$c_\pi(V) = c_\mu(V)$ for every $\mu\sigma-$open set V.
(b)$i_\pi(F) = i_\mu(F)$ for every $\mu\sigma-$closed set F.

Proof: (a) By Theorem 2.5(b) of [15], $c_\pi(V) = V \cup c_\mu i_\mu(V) = c_\mu i_\mu(V)$, since $V \in \sigma(\mu)$. Therefore, $c_\pi(V) = c_\mu(V)$.
(b) The proof follows from (a). □

Theorem 3.7. *Let (X,μ) be a quasi-topological space and $A\subset X$. If A is both a $b_\mu-$locally closed set and a $\mu\sigma-$open set, then A is a $\mu B-$set.*

Proof: Let A be both a $b_\mu-$locally closed set and a $\sigma-$open set. Then by Theorem 3.3, there exists an $\mu-$open set U such that $A=U\cap c_b(A)$ and so by Lemma 3.6, $A=U\cap(c_\sigma(A)\cap c_\pi(A))=U\cap(c_\sigma(A)\cap c_\mu(A))=U\cap c_\sigma(A)$. By Lemma 1.3, A is a $\mu B-$set. □

Theorem 3.8. *Let (X,μ) be a quasi-topological space. Then the following are equivalent.*
(a) A is a $\mu-$open set.
(b) A is both a $\mu\alpha-$open set and a $b_\mu-$locally closed set.

Proof: (a)⇒(b). Suppose that A is a $\mu-$open set. Since $\mu\subset\alpha(\mu)$, A is a $\mu\alpha-$open set. Since A is a $\mu-$open set and $A=A\cap X$, where X is a $\mu b-$closed set, A is a $b_\mu-$locally closed set.
(b)⇒(a). Suppose that A is both a $\mu\alpha-$open set and a $b_\mu-$locally closed set. Since $\alpha(\mu)\subset\sigma(\mu)$, A is a $\mu\sigma-$open set. Since A is a $b_\mu-$locally closed set, A is a $\mu B-$set by Theorem 3.7. Since $\alpha(\mu)\subset\pi(\mu)$, A is a $\mu\pi-$open set. Then by Lemma 1.2, A is a $\mu-$open set. □

Example 3.9. *(a) Consider the space in Example 3.2. Here, $\{a,d\}$ is a $b_\mu-$locally closed set, since $\{a,d\}=\{a,c,d\}\cap\{a,b,d\}$ and $\{a,d\}$ is not a $\mu\alpha-$open set since $i_\mu c_\mu i_\mu\{a,d\}=\{d\}$. Thus, $\{a,d\}$ is a $b_\mu-$locally closed set but not a $\mu\alpha-$open set.*
(b)Let $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{b,c,d\},\{b\}\}$. The family of all $\mu-$closed sets are $\{X,\{a\},\{a,c,d\}\}$. Here, $i_\mu c_\mu i_\mu\{b,c\}=\{b,c,d\}$ and so $\{b,c\}$ is a $\mu\alpha-$open set. Since $\{b,c,d\}$ is the only $\mu-$open set such that $\{b,c\}=\{b,c,d\}\cap\{b,c\}$ and $\{b,c\}$ is not a $\mu b-$closed set, $\{b,c\}$ is not a $b_\mu-$locally closed set.

A subset A of a space (X,μ) is said to be a $b-t-$*set* [3] if $i_\mu(A)=i_\mu(c_b(A))$. A is said to be a $b-B-$*set* [3] if $A=U\cap V$, where U is a $\mu-$open set and V is a $b-t-$set. A is said to be a $b-\sigma-$open [3](resp., $b-\pi-$open [3]) set if $A\subset c_\mu(i_b(A))$ (resp., $A\subset i_\mu(c_b(A))$). The following Remark 3.10, shows that the above notions are already defined notions in topological spaces.

Remark 3.10. *(a) By Theorem 3.8 (a) and (b) of [15], $i_\mu c_b(A)=c_b i_\mu(A)=i_\mu c_\mu i_\mu(A)$ and $c_\mu i_b(A)=i_b c_\mu(A)=c_\mu i_\mu c_\mu(A)$ for every subset A of X in a quasi-topological space. If (X,μ) is a quasi-topological space, then the family of all $b-t-$sets (resp., $b-B-$sets, $b-$semiopen sets, $b-$preopen sets) coincides with the family of all $\mu t^\star-$sets (resp., $\mu C-$sets, $\mu\beta-$open sets, $\mu\alpha-$open sets).*
(b) If (X,τ) is a topological space, then the family of all $b-t-$sets (resp., $b-B-$sets, $b-$semiopen sets, $b-$preopen sets) coincides with the family of all $\alpha^\star-$sets (resp., $C-$sets, $\beta-$open sets, $\alpha-$open sets). Hence Theorem 3.11(a) below is nothing but Proposition 3.1 of [10]. Theorem 3.11(c) is stated in Remark 3.2 of [10].

Theorem 3.11. *[3, Proposition 3.1] Let (X, τ) be a topological space and $A, B \subset X$. Then the following hold. .*
(a) A is a $b-t-$set if and only if it is a $b-$semiclosed set.
(b) If A is a $b-$closed set, then it is a $b-t-$set.
(c) If A and B are $b-t-$sets, then $A \cap B$ is a $b-t-$set.

The following Theorem 3.12 shows that the above Theorem 3.11 is valid, if *topology* is replaced by *generalized topology.*

Theorem 3.12. *Let (X, μ) be a space and $A, B \subset X$. Then the following hold.*
(a) A is a $b-t-$ set if and only if it is a $b-\sigma-$closed set.
(b) If A is a $\mu b-$closed set, then it is a $b-t-$set.

Proof: (a) Suppose that A is a $b-t-$set. Then $i_\mu(A) = i_\mu c_b(A)$ and $i_\mu c_b(A) \subset A$. Therefore, A is a $b-\sigma-$closed set. Conversely, if A is a $b-\sigma-$closed set then $i_\mu c_b(A) \subset A$ and $i_\mu c_b(A) \subset i_\mu(A)$. Since $A \subset c_b(A)$, $i_\mu(A) \subset i_\mu c_b(A)$. Hence $i_\mu(A) = i_\mu c_b(A)$. Therefore, A is a $b-t-$set.
(b) Let A be a $\mu b-$closed set. Then A is a $b-\sigma-$ closed set. Therefore, A is a $b-t-$set. □

By Remark 3.10(b), (a) of the following Theorem 3.13 is nothing but Proposition 3.2(a) of [10] and (b) is nothing but Proposition 3.4 of [10]. Moreover, this results hold for any generalized topological space as stated in Theorem 3.14 without proof.

Theorem 3.13. *[3, Proposition 3.2] Let (X, τ) be a topological space and $A \subset X$. Then the following hold.*
(a) A is a $t-$set then it is a $b-t-$set.
(b) If A is a $B-$set, then it is a $b-B-$set.

Theorem 3.14. *Let (X, μ) be a space and $A \subset X$. Then the following hold.*
(a) If A is a $\mu t-$set, then it is a $b-t-$set.
(b) If A is a $\mu B-$set then it is a $b-B-$set.

By Remark 3.10(b), the following Theorem 3.15 is nothing but Proposition 3.5 of [10]. The generalization of this theorem for quasi-topological spaces is established in Theorem 2.12 above.

Theorem 3.15. *[3, Theorem 3.1] Let (X, τ) be a topological space and $A \subset X$. Then the following are equivalent.*
(a)A is a open set.
(b)A is $b-$preopen and a $b-B-$set

By Remark 3.10(b), the following Theorem 3.16 is nothing but Proposition 3.2(c) of [10]. Moreover, this results hold for any generalized topological space as stated in Theorem 3.17 without proof.

Theorem 3.16. *[3, Proposition 3.3] Let (X, τ) be a topological space and $A \subset X$. Then the following are equivalent.*
(a)A is regular open.
(b)A is $b-$preopen and a $b-t-$set.

Theorem 3.17. *Let (X, μ) be a space and $A \subset X$. Then the following are equivalent.*
(a)A is a $\mu\pi-$regular set.
(b)A is both a $b-\pi-$open set and a $b-t-$set.

4. Decomposition of $(\mu, \mu')-$continuity

Let (X, μ) and (Y, μ') be two spaces. A mapping $f : X \to Y$ is said to be $(\mu, \mu')-$continuous [5] (resp. $(\mu\alpha, \mu')-$continuous, $(\mu B, \mu')-$continuous, $(\mu C, \mu')-$continuous, $(\mu b, \mu')-$continuous, $(D(\mu, b), \mu')-$continuous, $(\mu bLC, \mu')-$continuous) if for each $\mu'-$open set V, $f^{-1}(V)$ is a $\mu-$open set (resp. $\mu\alpha-$open set, $\mu B-$set, $\mu C-$ set, $\mu b-$open set, $D(\mu, b)-$set, $b_\mu-$locally closed set) in X. The following theorems give decompositions of $(\mu, \mu')-$continuity.

Theorem 4.1. *Let (X, μ) and (Y, μ') be two spaces where (X, μ) is a quasi-topological space and $f : X \to Y$ be a function. Then the following are equivalent.*
(a) f is $(\mu, \mu')-$continuous.
(b) f is $(\mu\alpha, \mu')-$continuous and $(\mu C, \mu')-$continuous.

Proof: The proof follows from Theorem 2.12. □

Theorem 4.2. *Let (X, μ) and (Y, μ') be two spaces and $f : X \to Y$ be a function. Then the following are equivalent.*
(a) f is $(\mu, \mu')-$continuous.
(b) f is $(\mu b, \mu')-$continuous and $(D(\mu, b), \mu')-$continuous.

Proof: The proof follows from Theorem 3.1. □

Theorem 4.3. *Let (X, μ) and (Y, μ') be two spaces where (X, μ) is a quasi-topological space and $f : X \to Y$ be a function. Then the following are equivalent.*
(a) f is $(\mu, \mu')-$continuous.
(b) f is $(\mu\alpha, \mu')-$continuous and $(\mu bLC, \mu')-$continuous.

Proof: The proof follows from Theorem 3.8. □

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