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Parallel Surfaces to Normal Ruled Surfaces of General Helices in the Sol Space $\mathfrak{Sol}^3$

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ABSTRACT: In this paper, parallel surfaces to normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are studied. Also, explicit parametric equations of parallel surfaces to normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are found.

Key Words: General helix, Sol Space, Normal surface, Parallel surface.

Contents

1. Introduction

Ruled surfaces have been popular in architecture. Structural elegance, these and many other contributions are in contrast to recent free-form architecture. Applied mathematics and in particular geometry have initiated the implementation of comprehensive frameworks for modeling and mastering the complexity of today's architectural needs shapes in an optimal sense by ruled surfaces.

In this paper, parallel surfaces to normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are studied. Also, explicit parametric equations of parallel surfaces to normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are found.

2. Riemannian Structure of Sol Space $\mathfrak{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{2.1}$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

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Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i, \tag{2.2}$$

where

$$\boldsymbol{\omega}^1 = e^z dx, \quad \boldsymbol{\omega}^2 = e^{-z} dy, \quad \boldsymbol{\omega}^3 = dz, \tag{2.3}$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.4}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric* $g_{\mathfrak{Sol}^3}$*, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i,j)*-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for the basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\mathfrak{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x,y,z) &\to (x+c, y, z), \\ (x,y,z) &\to (x, y+c, z), \\ (x,y,z) &\to \left(e^{-c}x, e^{c}y, z+c\right). \end{aligned}$$

3. General Helices in Sol Space $\mathfrak{Sol}^3$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{T}) &= 1,\ g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{N}) = 1,\ g_{\mathfrak{Sol}^3}(\mathbf{B},\mathbf{B}) = 1, \\ g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{B}) = g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, it is possible to write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. *([14]) Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{\sin\mathfrak{P}e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4, \\ y(s) &= \frac{\sin\mathfrak{P}e^{\cos\mathfrak{P}s+\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5, \\ z(s) &= \cos\mathfrak{P}s+\mathfrak{C}_3, \end{aligned} \tag{3.4}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.

An example of a graphic defined by Eq. (3.4) is illustrated in Fig. 1:

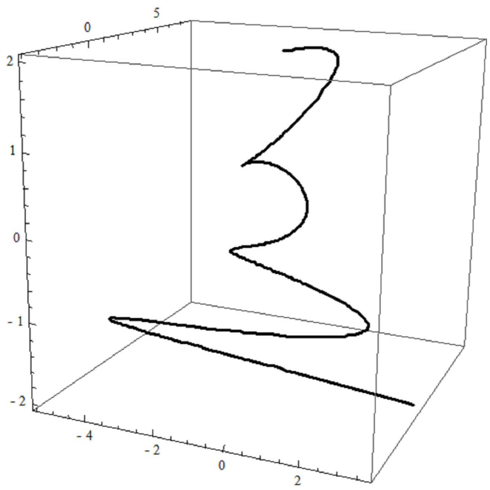

Fig. 1

4. Parallel Surfaces to Normal Ruled Surfaces of General Helices in $\mathfrak{Sol}^3$

The purpose of this section is to study parallel surfaces to normal ruled surfaces of general helices in $\mathfrak{Sol}^3$.

The normal ruled surface of γ is

$$\mathfrak{L}(s,u) = \gamma(s) + u\mathbf{N}. \tag{4.1}$$

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ is a unit speed non-geodesic general helix in $\mathfrak{Sol}^3$. Then, the equation of normal ruled surface of γ is*

$$\begin{aligned}\mathfrak{L}(s,u) = {} & [\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3} \\ & +\frac{\mathfrak{u}}{\kappa}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1 \\ & +[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3} \\ & +\frac{\mathfrak{u}}{\kappa}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2 \\ & +[\cos\mathfrak{P}s+\frac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_3]\mathbf{e}_3,\end{aligned} \tag{4.2}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.

A parallel surface to normal surface of γ is a parametrized surface

$$\mathcal{P}(s,u) = \mathfrak{L}(s,u) + \digamma\mathbf{n}_{\mathfrak{L}}(s,u), \tag{4.3}$$

where $\digamma$ is a constant.

Now, we can prove the following interesting main result.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix and $\mathfrak{L}$ its normal surface in Sol space. Then, equation of parallel surfaces to normal* *surface of γ is given by*

$$
\begin{aligned}
\mathcal{P} \quad = \quad & [[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3} \\
& +\frac{\mathfrak{u}}{\kappa}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& +\mathcal{F}[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1 \\
& +[[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3} \\
& +\frac{\mathfrak{u}}{\kappa}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \qquad (4.4) \\
& +\mathcal{F}[-\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2+[[\cos\mathfrak{P}s+\frac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_3]+\mathcal{F}[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& [\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]-\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& [-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]-\mathcal{N}\cos\mathfrak{P}]]\mathbf{e}_3,
\end{aligned}
$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration and

$$\mathcal{M}=\frac{1-u\kappa}{\sqrt{(1-u\kappa)^2+u^2\tau^2}},\ \mathcal{N}=\frac{u\tau}{\sqrt{(1-u\kappa)^2+u^2\tau^2}}.$$

Proof: From theorem 4.1, unit normal of normal surface of γ is

$$\begin{aligned}
\mathbf{n}_{\mathfrak{L}} = & [\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_1 \qquad (4.5) \\
& +[-\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_2 \\
& +[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\cos\mathfrak{P}]\mathbf{e}_3,
\end{aligned}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration and

$$\mathcal{M} = \frac{1-u\kappa}{\sqrt{(1-u\kappa)^2+u^2\tau^2}},\ \mathcal{N} = \frac{u\tau}{\sqrt{(1-u\kappa)^2+u^2\tau^2}}.$$

By using (4.2) and (4.3) we obtain (4.4). Hence the proof is completed. □

Theorem 4.3. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix and $\mathfrak{L}$ its normal surface in Sol space. Then, equations of parallel surfaces to normal surface of γ are given by*

$$\begin{aligned}
\boldsymbol{x} = & \exp[-[[\cos\mathfrak{P}s+\frac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_3] \\
& +\digamma[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\cos\mathfrak{P}]]][[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3} \\
& +\frac{\mathfrak{u}}{\kappa}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& +\digamma[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]],
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{y} \quad = \quad & \exp[[[\cos\mathfrak{P}s+\frac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1s+\mathfrak{C}_2]]+\mathfrak{C}_3] \\
& +\digamma[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2][\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2][-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\cos\mathfrak{P}]]][[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2]]+\mathfrak{C}_5e^{-\cos\mathfrak{P}s-\mathfrak{C}_3} \\
& +\frac{\mathfrak{u}}{\kappa}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2]]] \\
& +\digamma[-\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]]] \\
& -\mathcal{N}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2]]]],
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{z} \quad = \quad & [[\cos\mathfrak{P}s+\frac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1s+\mathfrak{C}_2]]+\mathfrak{C}_3] \\
& +\digamma[\mathcal{M}[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2][\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2][-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2] \\
& +\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]]]-\mathcal{N}\cos\mathfrak{P}]],
\end{aligned}
$$

where $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3,\mathfrak{C}_4,\mathfrak{C}_5$ *are constants of integration and*

$$
\mathcal{M}=\frac{1-u\kappa}{\sqrt{(1-u\kappa)^2+u^2\tau^2}},\ \mathcal{N}=\frac{u\tau}{\sqrt{(1-u\kappa)^2+u^2\tau^2}}.
$$

Proof: It is obvious from Theorem 4.2. □

We may use Mathematica, yields

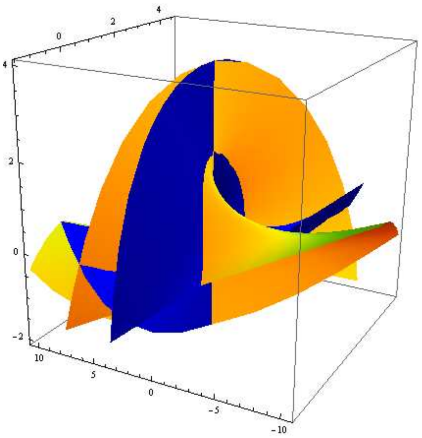

Fig. 2

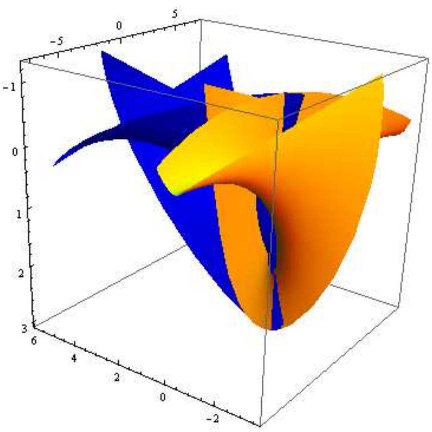

Fig. 3

Fig. 2,3: *The equations of normal surface and its parallel surface are illustrated colour Blue, Yellow, respectively.*

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