Bol. Soc. Paran. Mat. (3s.) **v. 31** 2 (2013): 235–244.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i2.17928

On the Spectrum of 2-nd order Generalized Difference Operator Δ^2 over the Sequence space c_0

S. Dutta and P. Baliarsingh

abstract: The main purpose of this article is to determine the spectrum and the fine spectrum of second order difference operator Δ^2 over the sequence space c_0. For any sequence $(x_k)_0^\infty$ in c_0, the generalized second order difference operator Δ^2 over c_0 is defined by $\Delta^2_k(x) = \sum_{i=0}^{2}(-1)^i\binom{2}{i}x_{k-i} = x_k - 2x_{k-1} + x_{k-2}$, with $x_k = 0$ for $k < 0$.

Key Words: Second order Difference operator; Spectrum of an operator; Sequence spaces.

Contents

1. Introduction, Preliminaries and Definitions

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $\mathcal{R}(T)$, we denote the range of T, i.e.

$$\mathcal{R}(T) = \{y \in Y : y = Tx\ ;\ x \in X\}.$$

By $B(X)$, we denote the set all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\mathbf{0}\}$ be a normed linear space over the complex field and $T : D(T) \to X$ be a linear operator, where $D(T)$ denotes the domain of T. With T, for a complex number λ, we associate an operator $T_\lambda = (T - \lambda I)$, where I is called identity operator on $D(T)$ and if T_λ has an inverse, we denote it by T_λ^{-1} i.e.

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and is called the *resolvent* operator of T. Many properties of T_λ and T_λ^{-1} depend on λ and the spectral theory is concerned with those properties. We are interested in the set of all λ's in the complex plane such that T_λ^{-1} exists/ T_λ^{-1} is bounded/ domain of T_λ^{-1} is dense in X. For our investigation, we need some basic concepts in spectral theory which are given as some definitions and lemmas.

2000 *Mathematics Subject Classification:* 47A10,40A05,46A45

Definition 1.1. *[12, pp. 371] Let X and T be defined as above. A regular value of T is a complex number λ such that*
(R1) *T_λ^{-1} exists;*
(R2) *T_λ^{-1} is bounded;*
(R3) *T_λ^{-1} is defined on a set which is dense in X.*

The *resolvent* set $\rho(T,X)$ of T is the set of all *regular* values of T. Its complement $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\rho(T,X)$ is partitioned into three disjoint sets as follows.
(I) ***Point spectrum*** $\sigma_p(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) does not hold. The elements of $\sigma_p(T,X)$ are called eigen values of T.
(II) ***Continuous spectrum*** $\sigma_c(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds and satisfies (R3) but does not satisfy (R2).
(III) ***Residual spectrum*** $\sigma_r(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds but does not satisfy (R3). The condition (R2) may or may not hold.
Goldberg's classification of operator T_λ :[9, pp. 58-71] Let X be a Banach space and $T_\lambda = (T - \lambda I) \in B(X)$, where λ is a complex number. Again, let $R(T_\lambda)$ and T_λ^{-1} denote the range and inverse of the operator T_λ respectively. Then the following possibilities may occur:
(A) $R(T_\lambda) = X$;
(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$;
(C) $\overline{R(T_\lambda)} \neq X$;
and
(1) T_λ is injective and T_λ^{-1} is continuous;
(2) T_λ is injective and T_λ^{-1} is discontinuous;
(3) T_λ is not injective.
Taking the permutations (A), (B), (C) and (1), (2), (3), we get nine different states. These are labelled by $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3. If λ is a complex number such that $T_\lambda \in A_1$ or $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T,X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda \in B_2\sigma(T,X)$ means the operator $T_\lambda \in B_2$, i.e. $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but $T_\lambda^{-1}$is discontinuous. Similarly others.

Lemma 1.2. *[9, pp. 59] A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 1.3. *[9, pp. 60] The adjoint operator T^* is onto if and and only if T has a bounded inverse.*

Let P, Q be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}_0$. For every $x = (x_k) \in P$ and every positive integer n, we write

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (P, Q)$ if and only if $Ax \in Q$ whenever $x \in P$.

Lemma 1.4. *[20, pp. 129] The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(i) *The rows of A are in ℓ_1 and their ℓ_1 norms are bounded,*

(ii) *The columns of A are in c_0,*

The operator norm of T is the supremum of ℓ_1 norms of the rows.

Lemma 1.5. *[20, pp. 126] The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Let ω be the set of all sequences of real or complex numbers. Any subspace of ω is called a sequence space and we write ℓ_∞, c, c_0 and ℓ_1 for the the set of linear spaces that are bounded, convergent, null and absolutely summable sequences respectively and for any sequence $x = (x_k)$ with the complex terms, normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

where $k \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, the set of non negative integers. Now, we define a second order difference operator $\Delta^2 : c_0 \to c_0$ by $\Delta^2(x) = \{\Delta_k^2(x)\}$, where

$$\Delta_k^2(x) = \sum_{i=0}^{2} (-1)^i \binom{2}{i} x_{k-i} = x_k - 2x_{k-1} + x_{k-2}, \tag{1.1}$$

with $x_k = 0$ for $k < 0$, where $x \in c_0$ and $k \in \mathbb{N}_0$. It is easy to verify that the operator Δ^2 can be represented by the matrix

$$\Delta^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The study of spectrum and fine spectrum of different operators carries a prominent and significant position in analysis. Various authors have their vigourous contributions in the study of spectral theory. In the existing literature, there are many articles concerning the spectrum and the fine spectra of an operator over different sequence spaces. For example; the fine spectrum of the Cesàro operator on the sequence space ℓ_p for $1 < p < \infty$ has been studied by Gonzalez [10]. The fine spectrum of the integer power of the Cesàro operator over c was examined by Wenger [19] and then Rhoades [16] generalized this result to the weighted mean methods. Reade [15] studied the spectrum of the Cesàro operator over the sequence space c_0. Okutoyi [14] computed the spectrum of the Cesàro operator over the sequence space bv. The fine spectra of the Cesàro operator over the sequence spaces c_0 and bv_p have been determined by Akhmedov and Başar [1,2]. Akhmedov and Başar [3,4] have studied the fine spectrum of the difference operator Δ over the sequence spaces c_0 and bv_p where $1 < p < \infty$. Altay and Başar [6,7] have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, for $0 < p < 1$. Furkan and Kayaduman [8] studied the fine spectrum of the difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv was investigated by Kayaduman and Furkan [11]. Srivastava and Kumar [17,18] have examined the fine spectrum of the generalized difference operator Δ_ν over the sequence spaces c_0 and ℓ_1. Recently, Panigrahi and Srivastava [13] studied the spectrum and fine spectrum of the generalized second order difference operator Δ^2_{uv} on c_0 and Akhmedov and Shabrawy [5] studied the fine spectrum of the operator $\Delta_{a,b}$ over c.

2. Main Results

In this section, we compute the spectrum, the point spectrum, the continuous spectrum, the residual spectrum and the fine spectrum of the operator Δ^2 on the sequence space c_0.

Theorem 2.1. *The operator $\Delta^2 : c_0 \to c_0$ is a linear operator and*

$$\| \Delta^2 \|_{(c_0:c_0)} = 4. \tag{2.1}$$

Proof: Proof of this theorem follows from the Lemma 1.5 and with the fact that

$$1 + |-2| + 1 = 4.$$

Theorem 2.2. *The spectrum of Δ^2 on the sequence space c_0 is given by*

$$\sigma(\Delta^2, c_0) = \Big\{ \alpha \in \mathbb{C} : |1-\alpha| \leq 3 \Big\}. \tag{2.2}$$

Proof: The proof of this theorem consists of two parts.
Part 1:
In the first part , we have to show that

$$\sigma(\Delta^2, c_0) \subseteq \Big\{ \alpha \in \mathbb{C} : |1-\alpha| \leq 3 \Big\}.$$

Equivalently, we need to show that if $\alpha \in \mathbb{C}$ with $|1-\alpha| > 3 \Rightarrow \alpha \notin \sigma(\Delta^2, c_0)$. Let $\alpha \in \mathbb{C}$ with $|1-\alpha| > 3$. Now $(\Delta^2 - \alpha I) = (a_{nk})$ is a triangle and hence has an inverse $(\Delta^2 - \alpha I)^{-1} = (b_{nk})$ where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{(1-\alpha)} & 0 & 0 & 0 & \cdots \\ \frac{2}{(1-\alpha)^2} & \frac{1}{(1-\alpha)} & 0 & 0 & \cdots \\ b_{20} & \frac{2}{(1-\alpha)^2} & \frac{1}{(1-\alpha)} & 0 & \cdots \\ b_{30} & b_{31} & \frac{2}{(1-\alpha)^2} & \frac{1}{(1-\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $b_{20}, b_{30}, b_{31}....$ etc. are as follows

$$b_{20} = \frac{2^2}{(1-\alpha)^3} - \frac{2}{2!(1-\alpha)^2},$$

$$b_{31} = \frac{2^2}{(1-\alpha)^3} - \frac{2}{2!(1-\alpha)^2},$$

$$\text{Similarly,} \quad b_{30} = \frac{2^3}{(1-\alpha)} - \frac{2^2}{(1-\alpha)^3}.$$

In general, for $k \in \mathbb{N}_0$ one can calculate

$$b_{kk} = \frac{1}{1-\alpha}, \quad b_{k,k-1} = \frac{2}{(1-\alpha)^2},$$

$$b_{k,k-2} = \frac{2^2}{(1-\alpha)^3} - \frac{2}{2!(1-\alpha)^2},$$

$$b_{k,k-3} = \frac{2^3}{(1-\alpha)^4} - \frac{2^2}{(1-\alpha)^3},$$

$$b_{k,k-4} = \frac{2^4}{(1-\alpha)^5} - \frac{2^3}{(1-\alpha)^4} - \frac{2^3}{2!(1-\alpha)^4} + \frac{2^2}{(2!)^2(1-\alpha)^3},$$

....

....

and so on.

Since $\frac{3}{|1-\alpha|} > \frac{2}{|1-\alpha|}$, it is clear that for each $k \in \mathbb{N}_0$, $|b_{nk}| < \infty$. Next to show that $(\Delta^2 - \alpha I)^{-1} \in (c_0, c_0)$, i.e.,

- the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$ and $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$,
- $\lim_{n\to\infty} |b_{nk}| = 0$ for each $k \in \mathbb{N}_0$.

Therefore, first we prove that the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$. Let

$$S_k = \sum_{j=0}^{k} |b_{kj}| = |b_{k0}| + |b_{k,1}| + |b_{k,2}| + ... + |b_{kk}| = |b_{kk}| + |b_{k,k-1}| + |b_{k,k-2}| + ...$$

$$= \left|\frac{1}{1-\alpha}\right| + \left|\frac{2}{(1-\alpha)^2}\right| + \left|\frac{2^2}{(1-\alpha)^3} - \frac{2}{2!(1-\alpha)^2}\right| + \left|\frac{2^3}{(1-\alpha)^4} - \frac{2^2}{(1-\alpha)^3}\right| + ...$$

Now,

$$
\begin{aligned}
\lim_k S_k &= \frac{1}{2}\left\{\left|\frac{2}{1-\alpha}\right| + \left|\frac{2^2}{(1-\alpha)^2}\right| + \left|\frac{2^3}{(1-\alpha)^3} - \frac{2^2}{2!(1-\alpha)^2}\right| + ...\right\} \\
&\leq \frac{1}{2}\left\{\left|\frac{2}{1-\alpha}\right| + \left|\frac{2}{1-\alpha}\right|^2 + \left|\frac{2}{1-\alpha}\right|^3 + \frac{1}{2}\left|\frac{2}{1-\alpha}\right|^2 + \left|\frac{2}{1-\alpha}\right|^4 + 2.\frac{1}{2}\left|\frac{2}{(1-\alpha)}\right|^3 + ...\right\} \\
&= \frac{1}{2}\left\{|\beta_2| + |\beta_2|^2 + |\beta_2|^3 + \frac{1}{2}|\beta_2|^2 + |\beta_2|^4 + 2.\frac{1}{2}|\beta_2|^3 + |\beta_2|^5 + 3.\frac{1}{2}|\beta_2|^4 + \frac{1}{4}|\beta_2|^3 + ...\right\} \\
&= \frac{1}{2}\left\{|\beta_2| + |\beta_2|^2\Big(\sum_{i\geq 1} n_{2(i)}\Big) + |\beta_2|^3\Big(\sum_{i\geq 1} n_{3(i)}\Big) + |\beta_2|^4\Big(\sum_{i\geq 1} n_{4(i)}\Big) + ...\right\} \\
&= \frac{1}{2}\left\{|\beta_2| + \Big(\frac{3}{2}\Big)|\beta_2|^2 + \Big(\frac{3}{2}\Big)^2|\beta_2|^3 + \Big(\frac{3}{2}\Big)^3|\beta_2|^4 + ...\right\} \\
&< \frac{1}{|1-\alpha|}\left\{1 + \left|\frac{3}{1-\alpha}\right| + \left|\frac{3}{1-\alpha}\right|^2 + \left|\frac{3}{1-\alpha}\right|^3 + ...\right\} \\
&= \frac{1}{|1-\alpha| - 3} < \infty.
\end{aligned}
$$

where $\beta_2 = \frac{2}{1-\alpha}$ and $n_{k(i)}$ denote the coefficients of $|\beta_2|^k$ for $k \geq 2$.

As per the assumption, $\left|\frac{3}{1-\alpha}\right| < 1$, hence $\lim_k S_k < \infty$. Now, (S_k) is a sequence of positive real numbers and is convergent, this implies the boundedness of (S_k).

Secondly, $\lim_{n\to\infty} |b_{nk}| = 0$, this follows from the fact that $\frac{3}{|1-\alpha|} > \frac{2}{|1-\alpha|}$ for each $k \in \mathbb{N}_0$. Thus, $(\Delta^2 - \alpha I)^{-1} \in B(c_0)$ with $|1-\alpha| > 3$.

Now, we show that domain of the operator $(\Delta^2 - \alpha I)^{-1}$ is dense in c_0 equivalently the range of $(\Delta^2 - \alpha I)$ is dense in c_0, which implies the operator $(\Delta^2 - \alpha I)^{-1}$ is onto. Hence we have

$$\sigma(\Delta^2, c_0) \subseteq \Big\{\alpha \in \mathbb{C} : |1-\alpha| \leq 3\Big\}. \tag{2.3}$$

Part 2:

Conversely, to show $\Big\{\alpha \in \mathbb{C} : |1-\alpha| \leq 3\Big\} \subseteq \sigma(\Delta^2, c_0)$. Consider $\alpha \neq 1$ and $|1-\alpha| < 3$, clearly $(\Delta^2 - \alpha I)$ is a triangle and hence $(\Delta^2 - \alpha I)^{-1}$ exists, but $\sup_k S_k$ is unbounded.

$$\Rightarrow \qquad (\Delta^2 - \alpha I)^{-1} \notin (c_0, c_0) \quad \text{with} \quad |1-\alpha| < 3.$$

Again $1 \neq \alpha \in \mathbb{C}$ with $|1-\alpha| = 3$ which implies $\lim_k S_k = \infty$. Thus $\sup_k S_k$ is unbounded.

$$\Rightarrow \qquad (\Delta^2 - \alpha I)^{-1} \notin (c_0, c_0) \quad \text{with} \quad |1-\alpha| = 3.$$

Finally we prove the result under the assumption $\alpha = 1$. We have

$$(\Delta^2 - \alpha I) = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ -2 & 0 & 0 & 0 & \dots \\ 1 & -2 & 0 & 0 & \dots \\ 0 & 1 & -2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is not invertible. Hence

$$\Big\{\alpha \in \mathbb{C} : |1-\alpha| \le 3 \Big\} \subseteq \sigma(\Delta^2, c_0). \tag{2.4}$$

Combining (2.3) and (2.4) we conclude the proof. □

Theorem 2.3. *Point spectrum of the operator Δ^2 over c_0 is given by*

$$\sigma_p(\Delta^2, c_0) = \emptyset.$$

Proof: Suppose $x \in c_0$ and consider $\Delta^2 x = \alpha x$ for $x \neq \theta$ in c_0, which gives a system of linear equations:

$$\left.\begin{array}{c} x_0 = \alpha x_0 \\ -2x_0 + x_1 = \alpha x_1 \\ x_0 - 2x_1 + x_2 = \alpha x_2 \\ x_1 - 2x_2 + x_3 = \alpha x_3 \\ \dots\dots \\ x_{k-2} - 2x_{k-1} + x_k = \alpha x_k \\ \dots\dots \end{array}\right\} \tag{2.5}$$

On solving above system of equations, it is clear that $\alpha = 1$ and $x_0 = 0, x_1 = 0, x_2 = 0, ...$ which contradicts our assumption.

Again suppose x_{ko} is the first non zero entry of $x = (x_k)$ and from the above system of equations $x_1 = \frac{2}{1-\alpha}x_0$, similarly we get $x_2 = [\frac{2^2}{(1-\alpha)^2} - \frac{2}{2(1-\alpha)}]x_0$ and so on. Proceeding this way, we can get $(\Delta^2 - \alpha I)x = \mathbf{0}$ has a solution for $\alpha \neq 1$ and $x_{ko-1} \neq 0$, which is a contradiction. Thus $\sigma_p(\Delta^2, c_0) = \emptyset$. □

Theorem 2.4. *Point spectrum of the dual operator $(\Delta^2)^*$ of Δ^2 over $c_0^* \cong \ell_1$ is given by*

$$\sigma_p((\Delta^2)^*, \ell_1) = \Big\{\alpha \in \mathbb{C} : |1-\alpha| \le 2 \Big\}. \tag{2.6}$$

Proof: Suppose $(\Delta^2)^* f = \alpha f$ for $\mathbf{0} \neq f \in \ell_1$, where

$$(\Delta^2)^* = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ 0 & 0 & 1 & -2 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

Consider the system of linear equations

$$\left.\begin{array}{c} f_0 - 2f_1 + f_2 = \alpha f_0 \\ f_1 - 2f_2 + f_3 = \alpha f_1 \\ f_2 - 2f_3 + f_4 = \alpha f_2 \\ \dots\dots \\ f_k - 2f_{k+1} + f_{k+2} = \alpha f_k \\ \dots\dots \end{array}\right\} \tag{2.7}$$

Since $\frac{2}{|1-\alpha|} > 1$, we have

$$\begin{aligned} |f_k| =& \frac{2}{|1-\alpha|}\left|\left(f_{k+1} - \frac{1}{2}f_{k+2}\right)\right| \\ >& \left|f_{k+1} - \frac{1}{2}f_{k+2}\right| \geq |f_{k+1}| - \left|\frac{1}{2}f_{k+2}\right|. \end{aligned}$$

It is clear that for each $k \in \mathbb{N}_0$, $f = (0, 0, ...f_k, f_{k+1}, 0, ...)$ is an eigen vector corresponding to λ satisfying $|1-\alpha| \leq 2$. On the basis of similar techniques one can show $|f_0| \geq |f_1| \geq |f_2| ... \geq |f_k| ...$ i.e. $|f_0| \geq |f_k|$, for $k \in \mathbb{N}_0$, this implies that $\sup_k |f_k| < \infty$ provided $|f_0| < \infty$.
Conversely, it is trivial to show that if $\sup_k |f_k| < \infty$, then $|1-\alpha| \leq 2$. □

Theorem 2.5. *Residual spectrum of the operator Δ^2 over c_0 is given by*

$$\sigma_r(\Delta^2, c_0) = \Big\{\alpha \in \mathbb{C} : |1-\alpha| \leq 2 \Big\}. \tag{2.8}$$

Proof: For $|1-\alpha| < 2$, the operator $\Delta^2 - \alpha I$ has an inverse. By Theorem 2.4 the operator $(\Delta^2)^* - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ with $|1-\alpha| \leq 2$. By using Lemma 1.2, we have $\overline{R(\Delta^2 - \alpha I)} \neq c_0$. Hence

$$\sigma_r(\Delta^2, c_0) = \Big\{\alpha \in \mathbb{C} : |1-\alpha| \leq 2 \Big\}.$$

□

Theorem 2.6. *Continuous spectrum of the operator Δ^2 over c_0 is given by*

$$\sigma_c(\Delta^2, c_0) = \Big\{\alpha \in \mathbb{C} : 2 < |1-\alpha| \leq 3 \Big\}. \tag{2.9}$$

Proof: The proof of this theorem follows from Theorems 2.2, 2.3, 2.5 and the fact that

$$\sigma(\Delta^2, c_0) = \sigma_p(\Delta^2, c_0) \cup \sigma_r(\Delta^2, c_0) \cup \sigma_c(\Delta^2, c_0).$$

□

Theorem 2.7. *If α satisfies $|1-\alpha| > 2$, then $(\Delta^2 - \alpha I) \in A_1$.*

Proof: Suppose $\alpha \in \mathbb{C}$ with $|1-\alpha| > 2$, which implies $\alpha \neq 1$. Therefore, the operator $(\Delta^2 - \alpha I)$ is a triangle and hence has an inverse $(\Delta^2 - \alpha I)^{-1}$. This implies the operator $(\Delta^2 - \alpha I)^{-1}$ is continuous with $|1-\alpha| > 2$,
Now, let

$$\begin{aligned} (\Delta^2 - \alpha I)x = y \quad &\Rightarrow \quad x = (\Delta^2 - \alpha I)^{-1} y. \\ &\Rightarrow \quad x_k = ((\Delta^2 - \alpha I)^{-1} y)_{k=0}^{\infty}, \quad k \in \mathbb{N}_0. \end{aligned}$$

For every $y \in c_0$, we have a corresponding $x \in c_0$ such that $(\Delta^2 - \alpha I)x = y$. This shows that the operator $(\Delta^2 - \alpha I)$ is onto. □

Theorem 2.8. *If $\alpha \neq 1$ and $\alpha \in \sigma_r(\Delta^2, c_0)$, then $\alpha \in C_2\sigma(\Delta^2, c_0)$.*

Proof: By Theorem 2.4, the operator $(\Delta^2)^* - \alpha I$ is not one to one. Combining Lemma 1.2 with Theorem 2.4 we have the operator $\Delta^2 - \alpha I$ does not have dense range, which implies $\Delta^2 - \alpha I \in C$.

For the second part, since $\alpha \neq 1$ the operator $\Delta^2 - \alpha I$ has an inverse. As $\alpha \in \sigma_r(\Delta^2, c_0)$, this implies that $|1 - \alpha| \leq 2$, hence the operator $(\Delta^2 - \alpha I)^{-1}$ is discontinuous with $|1 - \alpha| \leq 2$. Therefore, $(\Delta^2 - \alpha I)^{-1} \in 2$. This completes the proof. □

Theorem 2.9. *If $\alpha \in \sigma_c(\Delta^2, c_0)$, then $\alpha \in B_2\sigma(\Delta^2, c_0)$.*

Proof: For $\alpha \in \sigma_c(\Delta^2, c_0)$, by Theorem 2.2 (Part 2) the operator $(\Delta^2 - \alpha I)^{-1}$ is discontinuous, which implies $(\Delta^2 - \alpha I)^{-1} \in 2$.

Furthermore, for $\alpha \in \sigma_c(\Delta^2, c_0)$ the operator $((\Delta^2)^* - \alpha I)$ is one to one. By Lemma 1.2 $\overline{R(\Delta^2 - \alpha I)} = c_0$. This completes the proof. □

Acknowledgments

We are very much grateful to the referees for their valuable suggestions and comments, which improved the presentation of this article.

References

1. A. M. Akhmedov, F. Başar, On the spectrum of the Cesàro operator in c_0, Math. J. Ibaraki Univ., **36**(2004), 25-32.
2. A. M. Akhmedov, F. Başar, The fine spectra of Cesàro operator C_1 over the sequence space bv_p, Math. J. Okayama Univ., **50**(2008), 135-147.
3. A. M. Akhmedov, F. Başar, The fine spectra of the difference operator Δ over the sequence space $\ell_p, (1 \leq p < \infty)$, Demonstratio Math., **39(3)**(2006), 586-595.
4. A. M. Akhmedov, F. Başar, On the fine spectra of the difference operator Δ over the sequence space $bv_p, (1 \leq p < \infty)$, Acta. Math. Sin. Engl. ser. Oct., **23(10)**(2007), 1757-1768.
5. A. M. Akhmedov, S. R. El-Shabrawy, On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c , Comput. Math. Appl., **61**(2011), 2994-3002.
6. B. Altay, F. Başar, The fine spectrum and the matrix domain of the difference operator Δ on the sequence space $\ell_p, (0 < p < 1)$, Comm. in Math. Anal., **2**(2007), 1-11.
7. B. Altay, F. Başar, On the fine spectrum of the difference operator on c_0 and c, Inform. Sci., **168**(2004), 217-224.
8. H. Furkan, H. Bilgiç, K. Kayaduman, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_1 and bv, Hokkaido Math. J., **35(4)**(2006), 893-904.
9. S. Goldberg, Unbounded linear operators, Dover Publications, Inc. New York, 1985.
10. M. Gonzalez, The fine spectrum of the Cesàro operator in $\ell_p, (1 < p < \infty)$, Arch. Math., **44**(1985), 355-358.
11. K. Kayaduman, H. Furkan, The fine spectra of the difference operator Δ over the sequence spaces ℓ_1 and bv , Int. Math. For., **1(24)**(2006), 1153-1160.

12. E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons Inc. New York-Chichester -Brisbane-Toronto, 1978.

13. B. L. Panigrahi, P. D. Srivastava, Spectrum and fine spectrum of the generalized second order difference operator Δ^2_{uv} on sequence space c_0, Thai. J. Math., **9(1)**(2011), 57-74.

14. J. T. Okutoyi, On the spectrum of C_1 as an operator on bv, Commun. Fac. Sci. Univ. Ank., Ser., $A_1$41 (1992), 197-207.

15. J. B. Reade, On the spectrum of the Cesàro operator, Bull. Lond. Math. Soc., **17**(1985), 263-267.

16. B. E. Rhoades, The fine spectra for weighted mean operators, Pacific J. Math., **104(1)**(1983), 219-230.

17. P. D. Srivastava, S. Kumar, On the fine spectrum of the generalized difference operator Δ_ν over the sequence space c_0, Commun. Math. Anal., **6(1)**(2009), 8-21.

18. P. D. Srivastava, S. Kumar, Fine spectrum of the generalized difference operator Δ_ν on sequence space ℓ_1, Thai. J. Math., **8(2)**(2010), 221-233.

19. R. B. Wenger, The fine spectra of Holder summability operators, Indian J. Pure Appl. Math., **6**(1975), 695-712.

20. A. Wilansky, Summability through Functional Analysis, North-Holland Mathematics Studies, Amsterdam, New York, Oxford, 1984.

S. Dutta
Department of Mathematics,
Utkal University,
Vanivihar, Bhubaneswar, India
E-mail address: saliladutta516@gmail.com

and

P. Baliarsingh
Department of Mathematics,
Trident Academy of Technology,
Infocity, Bhubaneswar-751024, India
E-mail address: pb.math10@gmail.com