Heredity for triangular operators

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ABSTRACT: A proof is given that if the lower triangular infinite matrix $T$ acts boundedly on $\ell^2$ and $U$ is the unilateral shift, the sequence $(U^*)^nTU^n$ inherits from $T$ the following properties: posinormality, dominance, $M$-hyponormality, hyponormality, normality, compactness, and noncompactness. Also, it is demonstrated that the upper triangular matrix $T^*$ is dominant if and only if $T$ is a diagonal matrix.

Key Words: posinormal operator, dominant operator, compact operator, $M$-hyponormal operator, hyponormal operator, triangular matrix, terraced matrix

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1. Introduction

In this paper we extend the results of [2] to a much larger collection of operators, and we include more properties of those operators.

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space $H$, then $A \in B(H)$ is said to be is posinormal if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, and $A$ is dominant if $\text{Ran}(A - \lambda) \subset \text{Ran}(A - \lambda)^*$ for all $\lambda$ in the spectrum of $A$. In [1] it is shown that $A$ is dominant if and only if $A - \lambda$ is posinormal for each complex number $\lambda$. The operator $A \in B(H)$ is $M$-hyponormal (see [5]) if there exists an $M > 0$ such that $\|((A - \lambda)^*f)^2 \leq M\|((A - \lambda)f)^2$ for all complex numbers $\lambda$ and all $f \in \ell^2$; if the inequality is satisfied for $M = 1$ and $\lambda = 0$, then $A$ is hyponormal. Hyponormal and $M$-hyponormal operators are necessarily dominant.

We note that if $T = [t_{ij}]$ is a lower triangular infinite matrix and $U$ is the unilateral shift, then $U^*TU$ is the lower triangular infinite matrix that is obtained when the first row and first column are deleted from $T$. The lower triangular matrix $T$ is terraced if its row segments are constant. In an earlier paper [2] it was shown that the hyponormality of a terraced matrix $T \in B(\ell^2)$ is inherited by $U^*TU$. Here, we will observe that because of a key technical lemma, a similar
result holds for lower triangular infinite matrices in general, and that an analogous result also holds for other properties: posinormality, dominance, $M$-hyponormality, normality, compactness, and noncompactness.

2. Main Results

The following lemma will play a key role in the proof of the main theorem.

**Lemma 2.1.** Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and $U$ is the unilateral shift. If $Y$ is any infinite matrix that acts boundedly on $\ell^2$, then

$$U^*(TY)U = (U^*TU)(U^*YU).$$

**Proof:** First we calculate the entries of $T^*Y$, obtaining

$$[T^*Y]_{ij} = \sum_{k=1}^{\infty} t_{ki}y_{kj}.$$ 

If we then delete the first row and first column from $T^*Y$, the entries of the resulting matrix $U^*(T^*Y)U$ agree with the corresponding entries of $(T^*)'Y'$ where $Y' = U^*YU$. 

In contrast, since $t_{10}y_{01} + t_{11}y_{11} \neq t_{11}y_{11}$ when $t_{10}y_{01} \neq 0$, we have a different result for $TY$:

$$U^*(TY)U \neq (U^*TU)(U^*YU).$$

We are now ready for the main result.

**Theorem 2.2.** Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and $U$ is the unilateral shift. Then $T' := U^*TU$ inherits each of the following properties from $T$:

(a) posinormality;
(b) dominance;
(c) $M$-hyponormality;
(d) hyponormality;
(e) normality;
(f) compactness; and
(g) noncompactness.

**Proof:**

(a) Suppose $T$ is posinormal. By [1, Theorem 2.1], $T = T^*B$ for some operator $B \in B(\ell^2)$. Now we apply Lemma 2.1 to obtain

$$T' = U^*TU = U^*(T^*B)U = (U^*TU)(U^*BU) = (T')^*(U^*BU);$$

by another application of [1, Theorem 2.1], $T'$ is posinormal.
(b) Assume that \( T \) is dominant. This means that \( T - \lambda \) is posinormal for all complex numbers \( \lambda \), so \( T - \lambda = (T - \lambda)^* B(\lambda) \) for some operator \( B(\lambda) \in B(\ell^2) \).

Since \( T - \lambda \) is lower triangular, Lemma 2.1 once again applies:
\[
T' - \lambda = U^* T U - \lambda = U^*(T - \lambda) U = U^* ( (T - \lambda)^* B(\lambda) ) U
\]
\[
= (U^* (T - \lambda)^* U) (U^* B(\lambda) U) = (T' - \lambda)^* (U^* B(\lambda) U),
\]
so again by [1, Theorem 2.1], \( T' - \lambda \) is posinormal. This is true for all \( \lambda \), so \( T' \) is dominant.

(c) Suppose that \( T \) is \( M \)-hyponormal, so for some \( M > 0 \) and all \( f \in \ell^2 \),
\[
M(\langle (T - \lambda)^*(T - \lambda)f, f \rangle - \langle (T - \lambda)(T - \lambda)^* f, f \rangle) \geq 0
\]
for all complex numbers \( \lambda \). Besides Lemma 2.1, we will use the fact that the unilateral shift \( U \) satisfies \( I -UU^* \geq 0 \). Then
\[
M(\langle (T' - \lambda)^*(T' - \lambda)f, f \rangle - \langle (T' - \lambda)(T' - \lambda)^* f, f \rangle) =
M(\langle U^* (T - \lambda)^* U(U^* (T - \lambda) U)f, f \rangle - \langle (U^* (T - \lambda) U)(U^* (T - \lambda) U)^* f, f \rangle) =
M(\langle (T - \lambda)^*(T - \lambda)f, f \rangle - \langle (T - \lambda)(T - \lambda)^* f, f \rangle
+ \langle (T - \lambda)(T - \lambda)^* U f, f \rangle)
= M(\langle (T - \lambda)^*(T - \lambda)f, f \rangle - \langle (T - \lambda)(T - \lambda)^* U f, f \rangle
+ \langle (I -UU^*)(T - \lambda)^*U f, (T - \lambda)^*U f \rangle) \geq 0
\]
for all \( f \) and all \( \lambda \), and therefore \( T' \) is \( M \)-hyponormal.

(d) This proof involves only a minor modification of the preceding argument with \( M = 1 \) and \( \lambda = 0 \).

(e) If \( T \) is normal, then \( TT^* = T^*T \), so it is not hard to show that \( T \) must be a diagonal matrix. Consequently, \( T' \) is also a diagonal matrix and thus also normal.

(f) Trivial.

(g) We prove the contrapositive. Assume that \( T' \) is compact, so \( UT'U^* \) is also compact. We note that \( T - UT'U^* \) has nonzero entries only in the first column. Since \( T \) is bounded, we must have \( \sum_{i=0}^{\infty} |t_{i0}|^2 = \|Te_0\|^2 < \infty \), where \( e_0 \) belongs to the standard orthonormal basis for \( \ell^2 \); consequently, \( T - UT'U^* \) is a Hilbert-Schmidt operator on \( \ell^2 \) and is therefore compact. Thus
\[
T = UT'U^* + (T - UT'U^*)
\]
is compact, since it is the sum of two compact operators.

If \( n \) is a positive integer, then \( (U^*)^nTU^n \) is obtained by deleting the first \( n \) rows and the first \( n \) columns from \( T \).

Corollary 2.3. For each positive integer \( n \), \( (U^*)^nTU^n \) inherits from \( T \) any of the properties (a) through (g) held by \( T \).

It is not hard to construct an example showing that non-normality is not inherited from a triangular operator \( T \), and that is left to the interested reader. Next we consider an example that helps settle other similar questions.
Example 2.4. Let $T$ denote the terraced matrix with row segments given by the sequence $\{r_n\}$ with $r_0 = 0$ and $r_n = 1/(n+1)$ for $n \geq 1$. If $f = e_0 - e_1$, then $f \in \text{Ker}T$ but $f \notin \text{Ker}T^*$, so by [1, Corollary 2.3], $T$ is not posinormal; consequently, $T$ is also not dominant, not $M$-hyponormal, and not hyponormal. But the terraced matrix $T'$ is hyponormal (see [1, Theorem 5.2] or [4]) and hence also $M$-hyponormal, dominant, and posinormal. So this example has shown that non-posinormality, non-dominance, non-$M$-hyponormality, and non-hyponormality are not inherited in general by triangular operators.

Finally, we turn now to the adjoint of a lower triangular operator $T$. The inheritance of compactness or noncompactness by $(T^*)'$ from the upper triangular matrix $T^*$ is clear. On the other hand, the question of whether posinormality is inherited in general by $(T^*)'$ from $T^*$ is unresolved, but the next result gives a nontrivial case in which posinormality is inherited.

Proposition 2.5. If $T$ is the terraced matrix associated with a positive decreasing sequence $\{r_n\}$ that converges to 0 and such that $\{(n+1)r_n\}$ is an increasing sequence that converges to $L < \infty$, then $T^*$ and $(T^*)'$ are both posinormal.

Proof: This result follows from [3, Corollary 3.1].

The next theorem shows that if $t_{ij} \neq 0$ for some $i > j$, then $T^*$ cannot be dominant; consequently, $T^*$ also cannot be hyponormal or $M$-hyponormal.

Theorem 2.6. If $T \in B(\ell^2)$ is a lower triangular infinite matrix and $t_{ij} \neq 0$ for some $i > j$, then $T^*$ is not dominant.

Proof: Let $j_0$ designate the first column in which $t_{i_0j_0} \neq 0$ for some $i_0 > j_0$. We find that $e_{j_0} \in \text{Ker}(T - t_{j_0j_0})^*$ but $e_{j_0} \notin \text{Ker}(T - t_{j_0j_0})$. Thus $(T - t_{j_0j_0})^*$ is not posinormal, so $T^*$ is not dominant. \qed

Thus we see that the adjoint $T^*$ of a lower triangular infinite matrix is dominant if and only if $T$ is a diagonal matrix; in that case, $(T^*)'$ trivially inherits posinormality, dominance, $M$-hyponormality, hyponormality, and normality from $T^*$.

References


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