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Heredity for triangular operators

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ABSTRACT: A proof is given that if the lower triangular infinite matrix T acts boundedly on ℓ^2 and U is the unilateral shift, the sequence $(U^*)^nTU^n$ inherits from T the following properties: posinormality, dominance, M-hyponormality, hyponormality, normality, compactness, and noncompactness. Also, it is demonstrated that the upper triangular matrix T^* is dominant if and only if T is a diagonal matrix.

Key Words: posinormal operator, dominant operator, compact operator, M-hyponormal operator, hyponormal operator, triangular matrix, terraced matrix

Contents

1. Introduction

In this paper we extend the results of [2] to a much larger collection of operators, and we include more properties of those operators.

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $A \in B(H)$ is said to be is *posinormal* if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, and A is *dominant* if

$$Ran(A-\lambda) \subset Ran(A-\lambda)^*$$

for all λ in the spectrum of A. In [1] it is shown that A is dominant if and only if $A-\lambda$ is posinormal for each complex number λ. The operator $A \in B(H)$ is *M-hyponormal* (see [5]) if there exists an $M > 0$ such that

$$||(A-\lambda)^*f||^2 \leq M||(A-\lambda)f||^2$$

for all complex numbers λ and all $f \in \ell^2$; if the inequality is satisfied for $M = 1$ and $\lambda = 0$, then A is *hyponormal*. Hyponormal and M-hyponormal operators are necessarily dominant.

We note that if $T = [t_{ij}]$ is a lower triangular infinite matrix and U is the unilateral shift, then U^*TU is the lower triangular infinite matrix that is obtained when the first row and first column are deleted from T. The lower triangular matrix T is *terraced* if its row segments are constant. In an earlier paper [2] it was shown that the hyponormality of a terraced matrix $T \in B(\ell^2)$ is inherited by U^*TU. Here, we will observe that because of a key technical lemma, a similar

2000 *Mathematics Subject Classification:* 47B99 result holds for lower triangular infinite matrices in general, and that an analogous result also holds for other properties: posinormality, dominance, M-hyponormality, normality, compactness, and noncompactness.

2. Main Results

The following lemma will play a key role in the proof of the main theorem.

Lemma 2.1. *Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and U is the unilateral shift. If Y is any infinite matrix that acts boundedly on ℓ^2, then*

$$U^*(T^*Y)U = (U^*T^*U)(U^*YU).$$

Proof: First we calculate the entries of T^*Y, obtaining

$$[T^*Y]_{ij} = \sum_{k=i}^{\infty} \overline{t_{ki}} y_{kj}.$$

If we then delete the first row and first column from T^*Y, the entries of the resulting matrix $U^*(T^*Y)U$ agree with the corresponding entries of $(T^*)'Y'$ where $Y' = U^*YU$. □

In contrast, since $t_{10}y_{01} + t_{11}y_{11} \neq t_{11}y_{11}$ when $t_{10}y_{01} \neq 0$, we have a different result for TY:

$$U^*(TY)U \neq (U^*TU)(U^*YU).$$

We are now ready for the main result.

Theorem 2.2. *Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and U is the unilateral shift. Then $T' :\equiv U^*TU$ inherits each of the following properties from T:*

(a) posinormality;
(b) dominance;
(c) M-hyponormality;
(d) hyponormality;
(e) normality;
(f) compactness; and
(g) noncompactness.

Proof:
(a) Suppose T is posinormal. By [1, Theorem 2.1], $T = T^*B$ for some operator $B \in B(\ell^2)$. Now we apply Lemma 2.1 to obtain

$$T' = U^*TU = U^*(T^*B)U = (U^*T^*U)(U^*BU) = (T')^*(U^*BU);$$

by another application of [1, Theorem 2.1], T' is posinormal.

(b) Assume that T is dominant. This means that $T-\lambda$ is posinormal for all complex numbers λ, so $T-\lambda=(T-\lambda)^*B(\lambda)$ for some operator $B(\lambda)\in B(\ell^2)$. Since $T-\lambda$ is lower triangular, Lemma 2.1 once again applies:

$$T'-\lambda=U^*TU-\lambda=U^*(T-\lambda)U=U^*((T-\lambda)^*B(\lambda))U$$

$$=(U^*(T-\lambda)^*U)(U^*B(\lambda)U)=(T'-\lambda)^*(U^*B(\lambda)U),$$

so again by [1, Theorem 2.1], $T'-\lambda$ is posinormal. This is true for all λ, so T' is dominant.

(c) Suppose that T is M-hyponormal, so for some $M>0$ and for all $f\in\ell^2$,

$$M\langle (T-\lambda)^*(T-\lambda)f,f\rangle-\langle (T-\lambda)(T-\lambda)^*f,f\rangle\geq 0$$

for all complex numbers λ. Besides Lemma 2.1, we will use the fact that the unilateral shift U satisfies $I-UU^*\geq 0$. Then

$$\begin{aligned}
&M\langle (T'-\lambda)^*(T'-\lambda)f,f\rangle-\langle (T'-\lambda)(T'-\lambda)^*f,f\rangle=\\
&M\langle (U^*(T-\lambda)^*U)(U^*(T-\lambda)U)f,f\rangle-\langle (U^*(T-\lambda)U)(U^*(T-\lambda)U)^*f,f\rangle=\\
&M\langle (U^*(T-\lambda)^*(T-\lambda)Uf,f\rangle-\langle (U^*(T-\lambda)U)(U^*(T-\lambda)U)^*f,f\rangle=\\
&M\langle (T-\lambda)^*(T-\lambda)Uf,Uf\rangle-\langle (T-\lambda)(T-\lambda)^*Uf,Uf\rangle\\
&\qquad +\langle (T-\lambda)(T-\lambda)^*Uf,Uf\rangle-\langle (T-\lambda)UU^*(T-\lambda)^*Uf,Uf\rangle=\\
&M\langle (T-\lambda)^*(T-\lambda)Uf,Uf\rangle-\langle (T-\lambda)(T-\lambda)^*Uf,Uf\rangle\\
&\qquad +\langle (I-UU^*)(T-\lambda)^*Uf,(T-\lambda)^*Uf\rangle\ \ \geq\ \ 0
\end{aligned}$$

for all f and all λ, and therefore T' is M-hyponormal.

(d) This proof involves only a minor modification of the preceding argument with $M=1$ and $\lambda=0$.

(e) If T is normal, then $TT^*=T^*T$, so it is not hard to show that T must be a diagonal matrix. Consequently, T' is also a diagonal matrix and thus also normal.

(f) Trivial.

(g) We prove the contrapositive. Assume that T' is compact, so $UT'U^*$ is also compact. We note that $T-UT'U^*$ has nonzero entries only in the first column. Since T is bounded, we must have $\sum_{i=0}^{\infty}|t_{i0}|^2=\|Te_0\|^2<\infty$, where e_0 belongs to the standard orthonormal basis for ℓ^2; consequently, $T-UT'U^*$ is a Hilbert-Schmidt operator on ℓ^2 and is therefore compact. Thus

$$T=UT'U^*+(T-UT'U^*)$$

is compact, since it is the sum of two compact operators. □

If n is a positive integer, then $(U^*)^nTU^n$ is obtained by deleting the first n rows and the first n columns from T.

Corollary 2.3. *For each positive integer n, $(U^*)^nTU^n$ inherits from T any of the properties (a) through (g) held by T.*

It is not hard to construct an example showing that non-normality is not inherited from a triangular operator T, and that is left to the interested reader. Next we consider an example that helps settle other similar questions.

Example 2.4. *Let T denote the terraced matrix with row segments given by the sequence $\{r_n\}$ with $r_0 = 0$ and $r_n = 1/(n+1)$ for $n \geq 1$. If $f = e_0 - e_1$, then $f \in KerT$ but $f \notin KerT^*$, so by [1, Corollary 2.3], T is not posinormal; consequently, T is also not dominant, not M-hyponormal, and not hyponormal. But the terraced matrix T' is hyponormal (see [1, Theorem 5.2] or [4]) and hence also M-hyponormal, dominant, and posinormal. So this example has shown that non-posinormality, non-dominance, non-M-hyponormality, and non-hyponormality are not inherited in general by triangular operators.*

Finally, we turn now to the adjoint of a lower triangular operator T. The inheritance of compactness or noncompactness by $(T^*)'$ from the upper triangular matrix T^* is clear. On the other hand, the question of whether posinormality is inherited in general by $(T^*)'$ from T^* is unresolved, but the next result gives a nontrivial case in which posinormality is inherited.

Proposition 2.5. *If T is the terraced matrix associated with a positive decreasing sequence $\{r_n\}$ that converges to 0 and such that $\{(n+1)r_n\}$ is an increasing sequence that converges to $L < \infty$, then T^* and $(T^*)'$ are both posinormal.*

Proof: This result follows from [3, Corollary 3.1]. □

The next theorem shows that if $t_{ij} \neq 0$ for some $i > j$, then T^* cannot be dominant; consequently, T^* also cannot be hyponormal or M-hyponormal.

Theorem 2.6. *If $T \in B(\ell^2)$ is a lower triangular infinite matrix and $t_{ij} \neq 0$ for some $i > j$, then T^* is not dominant.*

Proof: Let j_0 designate the first column in which $t_{i_0 j_0} \neq 0$ for some $i_0 > j_0$. We find that $e_{j_0} \in Ker(T - t_{j_0 j_0})^*$ but $e_{j_0} \notin Ker(T - t_{j_0 j_0})$. Thus $(T - t_{j_0 j_0})^*$ is not posinormal, so T^* is not dominant. □

Thus we see that the adjoint T^* of a lower triangular infinite matrix is dominant if and only T is a diagonal matrix; in that case, $(T^*)'$ trivially inherits posinormality, dominance, M-hyponormality, hyponormality, and normality from T^*.

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