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Weak forms of $\mathcal{G}$-α-open sets and decompositions of continuity via grills

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ABSTRACT: In this paper, we introduce and investigate the notions of weakly $\mathcal{G}$-semi-open sets and weakly $\mathcal{G}$-preopen sets in grill topological spaces. Furthermore, by using these sets we obtain new decompositions of continuity.

Key Words: grill topological space, $\mathcal{G}$-α-open set, weakly $\mathcal{G}$-semi-open set, weakly $\mathcal{G}$-preopen set, decomposition of continuity

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1. Introduction

The idea of grills on a topological space was first introduced by Choquet [6]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [4], [5], [13] for details). In [12], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [7] have defined new classes of sets in a grill topological space and obtained a new decomposition of continuity in terms of grills. In [2], the present authors defined and investigated the notions of $\mathcal{G}$-α-open sets, $\mathcal{G}$-semi-open sets and $\mathcal{G}$-β-open sets in topological space with a grill. By using these sets, we obtained decompositions of continuity. In this paper, we introduce and investigate the notions of weakly $\mathcal{G}$-semi-open sets and weakly $\mathcal{G}$-preopen sets in grill topological spaces. We introduce weakly $\mathcal{G}$-precontinuous functions and weakly $\mathcal{G}$-semi-continuous functions to obtain decompositions of continuity.

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2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$. A subcollection $\mathcal{G}$ of $\mathcal{P}(X)$ is called a grill [6] on X if $\mathcal{G}$ satisfies the following conditions:

1. $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,
2. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x.

Definition 2.1. *[12] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.*

Proposition 2.2. *[12] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$ the following properties hold:*

1. *$A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$,*
2. *$\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,*
3. *$\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$,*
4. *If $U \in \tau$, then $U \cap \Phi(A) \subseteq \Phi(U \cap A)$.*

Let $\mathcal{G}$ be a grill on a space X. Then we define a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\text{-}Cl(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Definition 2.3. *A subset A of a topological space X is said to be:*

1. *α-open [10] if $A \subseteq Int(Cl(Int(A)))$,*
2. *semi-open [8] if $A \subseteq Cl(Int(A))$,*
3. *preopen [9] if $A \subseteq Int(Cl(A))$,*
4. *β-open [1] or semi-preopen [3] if $A \subseteq Cl(Int(Cl(A)))$.*

Definition 2.4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A in X is said to be*

1. *Φ-open [7] if $A \subseteq Int(\Phi(A))$,*
2. *$\mathcal{G}$-α-open [2] if $A \subseteq Int(\Psi(Int(A)))$,*
3. *$\mathcal{G}$-preopen [7] if $A \subseteq Int(\Psi(A))$,*
4. *$\mathcal{G}$-semi-open [2] if $A \subseteq \Psi(Int(A))$,*
5. *$\mathcal{G}$-β-open [2] if $A \subseteq Cl(Int(\Psi(A)))$.*

The family of all $\mathcal{G}$-α-open (resp. $\mathcal{G}$-preopen, $\mathcal{G}$-semi-open, $\mathcal{G}$-β-open) sets in a grill topological space $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}\alpha O(X)$ (resp. $\mathcal{G}PO(X)$, $\mathcal{G}SO(X)$, $\mathcal{G}\beta O(X)$).

The following lemma is well-known.

Lemma 2.5. *For a subset A of a topological space (X, τ), the following properties hold:*

1. *$sCl(A) = A \cup Int(Cl(A))$,*
2. *$sCl(A) = Int(Cl(A))$ if A is open.*

Lemma 2.6. *For subsets A and B of a grill topological space $(X, \tau, \mathcal{G})$, the following properties hold:*

1. *$U \cap \Psi(A) \subseteq \Psi(U \cap A)$ if $U \in \tau$,*
2. *$Int(Cl(A)) \cap Int(Cl(B)) = Int(Cl(A \cap B))$ if either A or B is semi-open.*

Proof: (1) This follows from Proposition 2.2 (4).
(2) This follows from Lemma 3.5 of [11]. □

3. Weakly $\mathcal{G}$-semi-open sets

Definition 3.1. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be weakly $\mathcal{G}$-semi-open if $A \subseteq \Psi(Int(Cl(A)))$. The complement of a weakly $\mathcal{G}$-semi-open set is said to be weakly $\mathcal{G}$-semi-closed.*

Remark 3.2. *Every $\mathcal{G}$-semi-open set is weakly $\mathcal{G}$-semi-open but not conversely.*

Example 3.3. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$ and the grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$. Then $A = \{a\}$ is weakly $\mathcal{G}$-semi-open, but it is not $\mathcal{G}$-semi-open (also it is not semi-open).*

Proposition 3.4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then every weakly $\mathcal{G}$-semi-open set is β-open.*

Proof: Let A be a weakly $\mathcal{G}$-semi-open set. Then $A \subseteq \Psi(Int(Cl(A))) = \Phi(Int(Cl(A))) \cup Int(Cl(A)) \subseteq Cl(Int(Cl(A)))$. Hence A is β-open. □

The converse of Proposition 3.4 need not be true in general as shown in the following example.

Example 3.5. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and the grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$. Then $A = \{a, b\}$ is a semi-open set which is not weakly $\mathcal{G}$-semi-open.*

Remark 3.6. *The notions of weak $\mathcal{G}$-semi-openness and semi-openness are independent as shown by the above examples.*

Proposition 3.7. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A of $(X, \tau, \mathcal{G})$ is weakly $\mathcal{G}$-semi-closed if and only if $Int_{\mathcal{G}}(Cl(Int(A))) \subseteq A$.*

Proof: Let A be a weakly $\mathcal{G}$-semi-closed set of $(X, \tau, \mathcal{G})$, then $X - A$ is weakly $\mathcal{G}$-semi-open and hence $X - A \subseteq \Psi(Int(Cl(X - A))) = X - Int_{\mathcal{G}}(Cl(Int(A)))$. Therefore, we have $Int_{\mathcal{G}}(Cl(Int(A))) \subseteq A$.
Conversely, let $Int_{\mathcal{G}}(Cl(Int(A))) \subseteq A$. Then $X - A \subseteq \Psi(Int(Cl(X - A)))$ and hence $X - A$ is weakly $\mathcal{G}$-semi-open. Therefore, A is weakly $\mathcal{G}$-semi-closed. □

Since $\tau_{\mathcal{G}}$ is finer than τ, by Proposition 3.7, we have the following corollary.

Corollary 3.8. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If a subset A of $(X, \tau, \mathcal{G})$ is weakly $\mathcal{G}$-semi-closed, then $Int(\Psi(Int(A))) \subseteq A$.*

Theorem 3.9. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Let A, U and A_α $(\alpha \in \Delta)$ be subsets of X. Then the following properties hold:*

1. *If A_α is weakly $\mathcal{G}$-semi-open for each $\alpha \in \Delta$, then $\cup_{\alpha \in \Delta} A_\alpha$ is weakly $\mathcal{G}$-semi-open,*
2. *If U is open and A is weakly $\mathcal{G}$-semi-open, then $U \cap A$ is weakly $\mathcal{G}$-semi-open.*

Proof: (1) Since A_α is weakly $\mathcal{G}$-semi-open for each $\alpha \in \Delta$, we have $A_\alpha \subseteq \Psi(Int(Cl(A_\alpha)))$ for each $\alpha \in \Delta$ and by using Lemma 2.2, we obtain

$$\begin{aligned}
\cup_{\alpha \in \Delta} A_\alpha &\subseteq \cup_{\alpha \in \Delta} \Psi(Int(Cl(A_\alpha))) \\
&\subseteq \cup_{\alpha \in \Delta} \{\Phi(Int(Cl(A_\alpha))) \cup Int(Cl(A_\alpha))\} \\
&\subseteq \Phi(\cup_{\alpha \in \Delta} Int(Cl(A_\alpha))) \cup Int(Cl(\cup_{\alpha \in \Delta} A_\alpha)) \\
&\subseteq \Phi(Int(Cl(\cup_{\alpha \in \Delta} A_\alpha))) \cup Int(Cl(\cup_{\alpha \in \Delta} A_\alpha)) \\
&\subseteq \Psi(Int(Cl(\cup_{\alpha \in \Delta} A_\alpha))).
\end{aligned}$$

This shows that $\cup_{\alpha\in\Delta}A_\alpha$ is weakly $\mathcal{G}$-semi-open.
(2) Let A be weakly $\mathcal{G}$-semi-open and $U \in \tau$, then $A \subseteq \Psi(Int(Cl(A)))$ and by Lemma 2.2, we have

$$\begin{aligned} A\cap U &\subseteq \Psi(Int(Cl(A)))\cap U \\ &= [\Phi(Int(Cl(A)))\cup Int(Cl(A))]\cap U \\ &= [\Phi(Int(Cl(A)))\cap U]\cup[Int(Cl(A))\cap U] \\ &\subseteq \Phi[Int(Cl(A))\cap U]\cup[Int(Cl(A\cap U))] \\ &\subseteq \Phi(In(Cl(A\cap U)))\cup Int(Cl(A\cap U)) \\ &= \Psi(Int(Cl(A\cap U))). \end{aligned}$$

This shows that $A\cap U$ is weakly $\mathcal{G}$-semi-open. □

Remark 3.10. *A finite intersection of weakly $\mathcal{G}$-semi-open sets need not be weakly $\mathcal{G}$-semi-open in general as the following example shows.*

Example 3.11. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}\}$ and the grill $\mathcal{G} = \{\{a, c\}, \{a, c, b\}, \{a, c, d\}, X\}$. Then $A = \{b, d\}$ and $B = \{a, d\}$ are weakly $\mathcal{G}$-semi-open sets but $A \cap B$ is not weakly $\mathcal{G}$-semi-open.*

Definition 3.12. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a a $T_\mathcal{G}$-set if $A = U \cap V$, where $U \in \tau$ and $Int(V) = \Psi(Int(Cl(V)))$.*

Theorem 3.13. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset A of X, the following conditions are equivalent:*

1. *A is open;*
2. *A is weakly $\mathcal{G}$-semi-open and a $T_\mathcal{G}$-set.*

Proof: (1) ⇒ (2) Let A be any open set. Then we have $A = Int(A) \subseteq \Psi(Int(Cl(A)))$. Therefore A is weakly $\mathcal{G}$-semi-open and because $Int(X) = \Psi(Int(Cl(X)))$, thus A is a $T_\mathcal{G}$-set.
(2) ⇒ (1) If A is weakly $\mathcal{G}$-semi-open and also a $T_\mathcal{G}$-set, then $A \subseteq \Psi(Int(Cl(A))) = \Psi(Int(Cl(U \cap V)))$, where $U \in \tau$ and $Int(V) = \Psi(Int(Cl(V)))$. Hence

$$\begin{aligned} A \subseteq U\cap A &\subseteq U\cap\Psi(Int(Cl(U)))\cap\Psi(Int(Cl(V))) \\ &= U\cap Int(V) = Int(A). \end{aligned}$$

This shows that A is open. □

4. Weakly $\mathcal{G}$-preopen sets

Definition 4.1. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be weakly $\mathcal{G}$-preopen if $A \subseteq sCl(Int(\Psi(A)))$. The complement of a weakly $\mathcal{G}$-preopen set is said to be weakly $\mathcal{G}$-preclosed.*

Proposition 4.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then the following properties hold:*

1. *Every $\mathcal{G}$-preopen set is weakly $\mathcal{G}$-preopen,*
2. *Every weakly $\mathcal{G}$-preopen set is $\mathcal{G}$-β-open,*
3. *Every α-open set is weakly $\mathcal{G}$-preopen,*
4. *Every weakly $\mathcal{G}$-preopen set is preopen.*

Proof: (1) Let A be a $\mathcal{G}$-preopen set. Then we have

$$A \subseteq Int(\Psi(A)) \subseteq sCl[Int(\Psi(A))].$$

This shows that A is weakly $\mathcal{G}$-preopen.

(2) Let A be a weakly $\mathcal{G}$-preopen set. Then, since $sCl(B) \subseteq Cl(B)$ for any subset $B \subseteq X$, we have

$$A \subseteq sCl[Int(\Psi(A))] \subseteq Cl[Int(\Psi(A))].$$

This shows that A is $\mathcal{G}$-β-open.

(3) Let A be an α-open set. Then by Lemma 2.5, we have

$$\begin{aligned} A \subseteq & Int(Cl(Int(A))) \\ \subseteq & IntCl(Int(A \cup \Phi(A)))) \\ = & Int(Cl(Int(\Psi(A)))) \\ = & sCl(Int(\Psi(A))). \end{aligned}$$

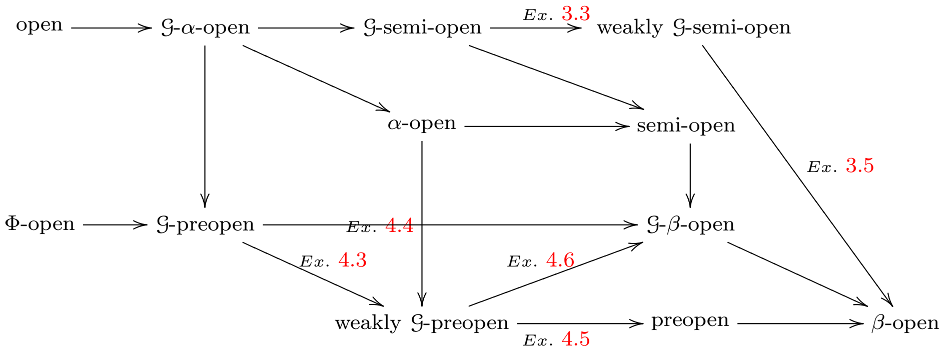
This shows that A is a weakly $\mathcal{G}$-preopen set.

(4) Let A be a weakly $\mathcal{G}$-preopen set. Then by Lemma 2.5, we have

$$\begin{aligned} A \subseteq & sCl(Int(\Psi(A))) \\ \subseteq & sCl(Int(Cl(A))) \\ = & Int(Cl(Int(Cl(A)))) \\ = & Int(Cl(A)). \end{aligned}$$

This shows that A is a preopen set. □

By Remark 2.5 of [2] and the results obtained above, the following diagram holds for a subset A of a space X.



Example 4.3. *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}, \{a, b, d\}\}$ *and the grill* $\mathcal{G} = \{\{a, c\}, \{a, c, b\}, \{a, c, d\}, X\}$. *Then* $A = \{a, b\}$ *is a weakly* $\mathcal{G}$*-preopen set which is not* $\mathcal{G}$*-preopen because* $\Psi(A) = A \cup \Phi(A) = A$, $sCl(Int(\Psi(A))) = X$. *But* $A \nsubseteq Int(\Psi(A)) = \{a\}$.

Example 4.4. *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}, \{a, b, c\}\}$ *and the grill* $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, \{d, c\}, \{a, c, b\}, \{a, c, d\}, \{b, c, d\}, X\}$. *Then* $A = \{c\}$ *is a weakly* $\mathcal{G}$*-preopen set which is not* α*-open because* $\Psi(A) = X$ *and* $Int(A) = \phi$.

Example 4.5. *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}, \{c, d\}\}$ *and the grill* $\mathcal{G} = \{\{a\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, c, b\}, \{a, c, d\}, \{a, b, d\}, X\}$. *Then* $A = \{b, c\}$ *is a pre-open set which is not weakly* $\mathcal{G}$*-preopen because* $sCl(Int(\Psi(A))) = \phi$ *and* $Cl(A) = X$.

Example 4.6. *Let* $X = \{a, b, c, d, e\}$, $\tau = \{\phi, X, \{a\}, \{b, e\}, \{a, b, e\}\}$ *and the grill* $\mathcal{G} = \{\{b, e\}, \{b, e, a\}, \{b, e, c\}, \{b, e, d\}, X\}$. *Then* $A = \{a, c\}$ *is a* $\mathcal{G}$*-*β*-open set which is not weakly* $\mathcal{G}$*-preopen because* $A \subseteq Cl(Int(\Psi(A))) = \{a, c, d\}$ *and* $A \nsubseteq sCl(Int(\Psi(A))) = \{a\}$.

Theorem 4.7. *Let* $(X, \tau, \mathcal{G})$ *be a grill topological space. Let* A, U *and* A_α $(\alpha \in \Delta)$ *be subsets of* X. *Then the following properties hold:*

1. *If* A_α *is weakly* $\mathcal{G}$*-preopen for each* $\alpha \in \Delta$, *then* $\cup_{\alpha\in\Delta} A_\alpha$ *is weakly* $\mathcal{G}$*-preopen.*
2. *If* U *is* α*-open and* A *is weakly* $\mathcal{G}$*-preopen, then* $U \cap A$ *is weakly* $\mathcal{G}$*-preopen.*

Proof: (1) Since A_α is weakly $\mathcal{G}$-preopen for each $\alpha \in \Delta$, we have

$$A_\alpha \subseteq sCl(Int(\Psi(A_\alpha))) \subseteq sCl\left(Int\left(\Psi\left(\cup_{\alpha\in\Delta} A_\alpha\right)\right)\right)$$

for each $\alpha \in \Delta$ and hence

$$\cup_{\alpha\in\Delta} A_\alpha \subseteq sCl\left(Int\left(\Psi\left(\cup_{\alpha\in\Delta} A_\alpha\right)\right)\right).$$

This shows that $\cup_{\alpha\in\Delta} A_\alpha$ is weakly $\mathcal{G}$-preopen.
(2) By Lemmas 2.5 and 2.6, we have

$$\begin{aligned} A\cap U &\subseteq sCl(Int(\Psi(A)))\cap Int(Cl(Int(U))) \\ &= Int(Cl(Int(\Psi(A))))\cap Int(Cl(Int(U))) \\ &= Int(Cl[Int(\Psi(A))\cap Int(U)]) \\ &= sCl(Int[\Psi(A)\cap Int(U)]) \\ &\subseteq sCl(Int(\Psi[A\cap Int(U)])) \\ &\subseteq sCl(Int(\Psi(A\cap U))). \end{aligned}$$

This shows that $U\cap A$ is weakly $\mathcal{G}$-preopen. □

Remark 4.8. *The finite intersection of weakly $\mathcal{G}$-preopen sets need not be weakly $\mathcal{G}$-preopen in general as the following example shows.*

Example 4.9. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}\}$ and the grill $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then $A = \{b, d\}$ and $B = \{a, d\}$ are weakly $\mathcal{G}$-preopen sets but $A\cap B$ is not weakly $\mathcal{G}$-preopen since $\Psi(A) = \Psi(B) = X$ and $sCl(Int(\Psi(A\cap B))) = \phi$.*

Theorem 4.10. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is weakly $\mathcal{G}$-preclosed if and only if $sInt(Cl(Int_{\mathcal{G}}(A))) \subseteq A$, where $Int_{\mathcal{G}}(A)$ denotes the interior of A with respect to $\tau_{\mathcal{G}}$.*

Proof: Let A be a weakly $\mathcal{G}$-preclosed set of $(X, \tau, \mathcal{G})$. Then $X - A$ is weakly $\mathcal{G}$-preopen and hence

$$X - A \subseteq sCl(Int(\Psi(X - A))) = X - sInt(Cl(Int_{\mathcal{G}}(A))).$$

Therefore, we have $sInt(Cl(Int_{\mathcal{G}}(A))) \subseteq A$.
Conversely, let $sInt(Cl(Int_{\mathcal{G}}(A))) \subseteq A$. Then

$$X - A \subseteq sCl(Int(\Psi(X - A)))$$

and hence $X - A$ is weakly $\mathcal{G}$-preopen. Therefore, A is weakly $\mathcal{G}$-preclosed. □

5. $A_{\mathcal{G}}$-sets and $N_{\mathcal{G}}$-sets

Definition 5.1. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}$-set [7] (resp. an $A_{\mathcal{G}}$-set, an $N_{\mathcal{G}}$-set) if $A = U\cap V$, where $U \in \tau$ and $Int(V) = Int(\Psi(V))$ (resp. $Int(V) = sCl(Int(\Psi(V)))$, $Int(V) = Cl(Int(\Psi(V)))$).*

Proposition 5.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset A of X, the following properties hold:*

1. *Every $N_{\mathcal{G}}$-set is an $A_{\mathcal{G}}$-set.*
2. *Every $A_{\mathcal{G}}$-set is a $\mathcal{G}$-set.*

Proof: (1) Let A be an $N_{\mathcal{G}}$-set. Then $A = U \cap V$, where $U \in \tau$ and $Int(V) = Cl(Int(\Psi(V)))$. Since $Int(V) = Cl(Int(\Psi(V))) \supseteq sCl(Int(\Psi(V)))$ and $Int(V) \subseteq Int(\Psi(V)) \subseteq sCl(Int(\Psi(V)))$, we have $Int(V) = sCl(Int(\Psi(V)))$ and hence $A = U \cap V$, where $U \in \tau$ and $Int(V) = sCl(Int(\Psi(V)))$. This shows that A is an $A_{\mathcal{G}}$-set.
(2) Let A be an $A_{\mathcal{G}}$-set. Then $A = U \cap V$, where $U \in \tau$ and $Int(V) = sCl(Int(\Psi(V)))$. Since $Int(\Psi(V)) \subseteq sCl(Int(\Psi(V))) = Int(V) \subseteq Int(\Psi(V))$, we obtain $Int(V) = Int(\Psi(V))$ and hence $A = U \cap V$, where $U \in \tau$ and $Int(V) = Int(\Psi(V))$. This shows that A is a $\mathcal{G}$-set.

□

The converses of the statements of Proposition 5.2 are not true as the following examples show.

Example 5.3. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{d\}, \{a, d\}\}$ and the grill $\mathcal{G} = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, d, b\}, \{b, d, c\}, \{c, d, a\}, X\}$. Then $A = \{a, b\}$ is an $A_{\mathcal{G}}$-set but which is not an $N_{\mathcal{G}}$-set. Since the only open set containing A is X, hence $A = X \cap V = V$ and $sCl(Int(\Psi(A))) = \{a\} = Int(A)$, but $Cl(Int(\Psi(A))) = \{a, b, c\} \neq Int(A)$.*

Example 5.4. *Let $X = \{a, b, c, d, e\}$, $\tau = \{\phi, X, \{a\}, \{b, e\}, \{a, b, e\}, \{a, c, d\}\}$ and the grill $\mathcal{G} = \{\{b, c\}, \{b, c, a\}, \{b, c, d\}, \{b, c, e\}, \{b, c, a, d\}, \{b, c, a, e\}, \{b, c, d, e\}, X\}$. Then $A = \{a, d\}$ is a $\mathcal{G}$-set but which is not an $A_{\mathcal{G}}$-set. Since the open set U containing A is X or $\{a, c, d\}$, hence $A = U \cap V$ and $Int(V) = Int(\Psi(V)) = \{a\}$ where V is any require subset of X. On the other hand, we have $sCl(Int(\Psi(V))) = \{a, c, d\} \neq Int(V)$.*

Theorem 5.5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset A of X, the following conditions are equivalent:*

1. *A is open;*
2. *A is weakly $\mathcal{G}$-preopen and an $A_{\mathcal{G}}$-set;*
3. *A is $\mathcal{G}$-β-open and an $N_{\mathcal{G}}$-set.*

Proof: (1) $\Rightarrow$ (2) Let A be any open set. Then we have $A = Int(A) \subseteq sCl(Int(\Psi(A)))$. Therefore A is weakly $\mathcal{G}$-preopen and because $Int(X) = sCl(Int(\Psi(X)))$, thus A is an $A_{\mathcal{G}}$-set.

(2) ⇒ (1) Let A be weakly $\mathcal{G}$-preopen and an $A_{\mathcal{G}}$-set. Let $A = U \cap C$, where U is open and $sCl(Int(\Psi(C))) = Int(C)$. Since A is a weakly $\mathcal{G}$-preopen set, we have

$$A \subseteq sCl(Int(\Psi(A))) = sCl(Int(\Psi(U \cap C))).$$

Hence

$$\begin{aligned} A \subseteq U \cap A &\subseteq U \cap [sCl(Int(\Psi(U))) \cap sCl(Int(\Psi(C)))] \\ &= U \cap Int(C) = Int(A). \end{aligned}$$

This shows that A is open.
(1) ⇒ (3) Let A be any open set. Then we have $A = Int(A) \subseteq Cl(Int(\Psi(A)))$. Therefore A is $\mathcal{G}$-β-open and because $Int(X) = Cl(Int(\Psi(X)))$, thus A is an $N_{\mathcal{G}}$-set.
(3) ⇒ (1) Let A be $\mathcal{G}$-β-open and an $N_{\mathcal{G}}$-set. Let $A = U \cap C$, where U is open and $Cl(Int(\Psi(C))) = Int(C)$. Since A is a $\mathcal{G}$-β-open set, we have

$$A \subseteq Cl(Int(\Psi(A))) = Cl(Int(\Psi(U \cap C))).$$

Hence

$$\begin{aligned} A \subseteq U \cap A &\subseteq U \cap [Cl(Int(\Psi(U))) \cap Cl(Int(\Psi(C)))] \\ &= U \cap Int(C) = Int(A). \end{aligned}$$

This shows that A is open. □

Remark 5.6. *The notion of weak $\mathcal{G}$-preopenness (resp. $\mathcal{G}$-β-openness) is independent of the notion of $A_{\mathcal{G}}$-sets (resp. $N_{\mathcal{G}}$-sets) as shown by the following examples.*

Example 5.7. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the grill $\mathcal{G} = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, d, b\}, \{b, d, c\}, \{c, d, a\}, X\}$. Then $A = \{a, c\}$ is an $A_{\mathcal{G}}$-set which is not weakly $\mathcal{G}$-preopen since $sCl(Int(\Psi(A))) = \{a\}$. If we take $\mathcal{G} = \mathcal{P}(X) - \{\phi\}$ in the same topology, then $B = \{a, d\}$ is a weakly $\mathcal{G}$-preopen set which is not an $A_{\mathcal{G}}$-set since $sCl(Int(\Psi(B))) = X$.*

Example 5.8. *(1) In Example 5.3, $A = \{a, b\}$ is $\mathcal{G}$-β-open, but it is not an $N_{\mathcal{G}}$-set.*
(2) In Example 5.7, if we take $\mathcal{G} = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, d, b\}, \{b, d, c\}, \{c, d, a\}, X\}$, then $A = \{c\}$ is an $N_{\mathcal{G}}$-set but it is not $\mathcal{G}$-β-open since $Cl(Int(\Psi(A))) = \phi$.

6. Decompositions of continuity

Definition 6.1. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $A_{\mathcal{G}}$-continuous (resp. $N_{\mathcal{G}}$-continuous, weakly $\mathcal{G}$-precontinuous, $\mathcal{G}$-β-continuous, weakly $\mathcal{G}$-semi-continuous, $T_{\mathcal{G}}$-continuous) if the inverse image of each open set of (Y, σ) is an $A_{\mathcal{G}}$-set (resp. an $N_{\mathcal{G}}$-set, weakly $\mathcal{G}$-preopen, $\mathcal{G}$-β-open, weakly $\mathcal{G}$-semi-open, a $T_{\mathcal{G}}$-set) in $(X, \tau, \mathcal{G})$.*

Theorem 6.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following conditions are equivalent:*

1. *f is continuous;*
2. *f is weakly $\mathcal{G}$-precontinuous and $A_{\mathcal{G}}$-continuous;*
3. *f is $\mathcal{G}$-β-continuous and $N_{\mathcal{G}}$-continuous;*
4. *f is weakly $\mathcal{G}$-semi-continuous and $T_{\mathcal{G}}$-continuous.*

Proof: This is an immediate consequence of Theorems 3.13 and 5.5. □

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