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Warfield p-Invariants in Abelian Group Rings of Characteristic p

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Abstract: We calculate Warfield p-invariants $W_{\alpha,p}(V(RG))$ of the group of normalized units $V(RG)$ in a commutative group ring RG of prime $char(RG) = p$ in each of the following cases:
(1) G_0/G_p is finite and R is an arbitrary direct product of indecomposable rings;
(2) G_0/G_p is bounded and R is a finite direct product of fields;
(3) $id(R)$ is finite (in particular, R is finitely generated).
Moreover, we give a general strategy for the computation of the above Warfield p-invariants under some restrictions on R and G. We also point out an essential incorrectness in a recent paper due to Mollov and Nachev in Commun. Algebra (2011).

Key Words: Abelian groups, commutative rings, indecomposable rings, units, Warfield p-invariants.

Contents

1. Introduction

Everywhere in the text, let R be a commutative unital ring of prime characteristic p and G an Abelian group written multiplicatively as is customary when discussing group rings. For such R and G, suppose RG is the group ring of G over R with unit group $U(RG)$ and its normalized component $V(RG)$; note that the decomposition $U(RG) = V(RG) \times U(R)$ holds, where $U(R)$ is the unit group (that is, the multiplicative group of units of R). As usual, $id(R) = \{e \in R \mid e^2 = e\}$ is the set of all idempotents of R.

Imitating [11], for any multiplicative group A we define the following ordinal-to-cardinal functions, called in the existing literature *Warfield p-invariants*

$$W_{\alpha,p}(A) = r(A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})),$$

where $\alpha \geq 0$ is an ordinal.

These invariants were the object of a series of explorations [1]- [6]. They were calculated for both $U(RG)$ and $V(RG)$ under some limitations on R and G only in their terms and divisions. The most important achievements are these:
(i) $G_0 = G_p$ (i.e., G is p-mixed) and R is arbitrary;
(ii) G_0/G_p is bounded and R is perfect;
(iii) G_0/G_p is bounded and R is a field;

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(iv) G_0/G_p is finite and R is indecomposable;

(v) G is arbitrary and R is perfect indecomposable.

Actually, the last result is proved in [1] for a perfect integral domain and in [2] for a perfect field, but according to the main theorem of [7] the same idea also works for an indecomposable ring.

Some other useful estimations of $W_{\alpha,p}(U(RG))$ and $W_{\alpha,p}(V(RG))$ are also obtained there.

Mollov and Nachev [10] have duplicated the results of ours from [1], [2], [3] and [4]. Even more, they have partly plagiarized results (i) and (v) as well as the ideas for their proofs without any concrete correct citation of the articles [2], [3] and [4].

Moreover, they wrongly cited in ([10], p.2300, the last sentence before Section 2) that [1] is the unique article of the current author which treated the problem for calculation of $W_{\alpha,p}(V(RG))$, but seeing the cited bibliography listed below this is apparently false.

The main purpose here is to add two more points to the list (i)-(v) given above, that are:

(vi) G_0/G_p is finite and R is an arbitrary direct product of indecomposable rings - thus extending (iv).

(vii) G_0/G_p is bounded and R is a finite direct product of fields - thus extending (iii).

We also give a general strategy for the computation of $W_{\alpha,p}(U(RG))$ over some special rings R.

2. Main Results

We first begin with a crucial technicality (see also [2]).

Lemma 2.1. *Let $A = \coprod_{i\in I} A_i$ be an abelian group. Then, for any ordinal α,*

$$W_{\alpha,p}(A) = \sum_{i\in I} W_{\alpha,p}(A_i).$$

Proof: Observe that for any ordinal β we have $A^{p^\beta} = \coprod_{i\in I} A_i^{p^\beta}$, and hence $A_p^{p^\beta} = \coprod_{i\in I}(A_i^{p^\beta})_p = \coprod_{i\in I}(A_i)_p^{p^\beta}$. Therefore,$A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha}) = \coprod_{i\in I}[A_i^{p^\alpha}/(A_i^{p^{\alpha+1}}(A_i)_p^{p^\alpha})]$, whence by a simple appeal to the additive property of the rank of an abelian group we derive that $r(A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})) = \sum_{i\in I} r(A_i^{p^\alpha}/(A_i^{p^{\alpha+1}}(A_i)_p^{p^\alpha}))$. The last is just equivalent to the desired equality. □

If $\{R_i\}_{i\in I}$ is a system of commutative unital rings for some finite or infinite index set I, then by $\prod_{i\in I} R_i$ we will denote the arbitrary direct product of rings in the following sense: Any element $r \in \prod_{i\in I} R_i$ is of the form of a vector (finite or infinite) $r = (\cdots, r_i, \cdots)$ equipped with the operations for an other element $f = (\cdots, f_i, \cdots)$ given by $r+f = (\cdots, r_i+f_i, \cdots)$ and $rf = (\cdots, r_if_i, \cdots)$. Clearly the zero element is $0 = (\cdots, 0_i, \cdots)$ where 0_i is the corresponding zero element in R_i, and the identity element is $1 = (\cdots, 1_i, \cdots)$ where 1_i is the corresponding identity element in R_i.

Under these circumstances, it is not difficult to check that $U(\coprod_{i\in I} R_i) = \coprod_{i\in I} U(R_i)$ which fact will be used in the sequel without a concrete referring.

Note that in some existing literature such a product is also called *a coproduct* of these rings R_i.

The next statement is well known but we will prove it for completeness and for the reader's convenience.

Proposition 2.2. *Let A be a finite group and let $K = \times_{j\in J} K_j$ be a finite direct product of rings. Then the following isomorphisms hold:*

$$(a)\ (\coprod_{i\in I} R_i)A \cong \coprod_{i\in I}(R_iA)$$

where I is an arbitrary index set.

$$(b)\ KG \cong \times_{j\in J}(K_jG).$$

Proof: (a) For any $v = \sum_{a\in A_v} r_a a$ where $r_a = (\cdots, r_{va}, \cdots) \in \coprod_{i\in I} R_i$ and A_v is a finite subset of A depending on the element v, define the map $\phi : (\coprod_{i\in I} R_i)A \to \coprod_{i\in I}(R_iA)$ via the equality $\phi(v) = (\cdots, \sum_{v\in A_v} r_{ai}a, \cdots)$. Furthermore, it is only a routine technical exercise to verify that ϕ is an isomorphism of R-algebras, as required.

(b) Follows in the same manner. □

Remark 2.1. *We will further identify with no loss of generality $(\coprod_{i\in I} R_i)A$ with $\coprod_{i\in I}(R_iA)$, and $(\times_{j\in J}K_j)G$ with $\times_{j\in J}(K_jG)$, so that the two isomorphisms in points (a) and (b) will be formal equalities, indeed.*

We are now ready to state and prove the following first main result.

Theorem 2.3. *Suppose G is a group whose factor G_0/G_p is finite and $R = \coprod_{i\in I} R_i$ where each R_i is indecomposable for $i \in I$. Then the following formula is valid:*

$$W_{\alpha,p}(U(RG)) = \mu\cdot W_{\alpha,p}(G) + \sum_{i\in I}\ \sum_{d/exp(G_0/G_p)} (l_d/(R_i(\zeta_d) : R_i))\cdot W_{\alpha,p}(U(R_i(\zeta_d))),$$

where $l_d = |\{a \in G_0/G_p\ :\ o(a) = d\}|$.

Proof: Since $\coprod_{q\neq p} G_q$ is finite and pure in G, one may write $G = (\coprod_{q\neq p} G_q) \times M$ for some p-mixed group M. Consequently, Proposition 2.2 (a) leads to $RG = [R(\coprod_{q\neq p} G_q)]M = [(\coprod_{i\in I} R_i)(\coprod_{q\neq p} G_q)]M = [\coprod_{i\in I} R_i(\coprod_{q\neq p} G_q)]M$. Furthermore, as in [6], $W_{\alpha,p}(U(RG)) = \mu\cdot W_{\alpha,p}(G) + W_{\alpha,p}(U(\coprod_{i\in I} R_i(\coprod_{q\neq p} G_q)))$ where

μ is given there explicitly. However, for each index $i \in I$, we have the relation $R_i(\coprod_{q\neq p} G_q) \cong \sum_{d/exp(\coprod_{q\neq p} G_q)}(l_d/(R_i(\zeta_d) : R_i))R_i(\zeta_d)$ (see, e.g., [9]). Thus, one may deduce that $U(\prod_{i\in I} R_i(\coprod_{q\neq p} G_q)) = \coprod_{i\in I} U(R_i(\coprod_{q\neq p} G_q)) \cong \coprod_{i\in I} \times_{d/exp(\coprod_{q\neq p} G_q)}(l_d/(R_i(\zeta_d) : R_i))U(R_i(\zeta_d))$ and henceforth Lemma 2.1 works. This gives the desired equalities. □

We are now in a position to formulate and prove

Theorem 2.4. *Let G be an abelian group for which G_0/G_p is infinite bounded and $R = F_1 \times \cdots \times F_n$ where every F_i is a field; $i \in [1, n]$, where n is natural. Then*

$$W_{\alpha,p}(U(RG)) = |id(R)| \cdot |\coprod_{q\neq p} G_q| \cdot W_{\alpha,p}(G) + \sum_{i=1}^{n} \coprod_{m=0}^{\infty} \coprod_{a_i(m)} W_{\alpha,p}(F_i(\zeta_m))$$

with $a_i(m) = |\{g \in \coprod_{q\neq p} G_q \ : \ o(g) = d\}|/(F_i(\zeta_m) : F_i)$.

Proof: Since $RG = F_1G \times \cdots \times F_nG$, we derive $U(RG) = U(F_1G) \times \cdots \times U(F_nG)$. Therefore, using Lemma 2.1, we deduce that $W_{\alpha,p}(U(RG)) = \sum_{i=1}^{n} W_{\alpha,p}(U(F_iG))$. Utilizing ([6], Theorem 2.2 (1)), $W_{\alpha,p}(U(F_iG))$ are completely computed, so that the wanted equality follows. □

Remark 2.2. *When G_0/G_p is finite bounded, things are settled in Theorem 2.3 listed above.*

The next statement somewhat supersedes Theorem 3.9 from [10].

Theorem 2.5. *Suppose R is a perfect ring with a finite number of idempotents (in particular, R is perfect finitely generated). Then the following formula holds:*

$$W_{\alpha,p}(U(RG)) = (\sum_{k=1}^{n} \sum_{d/k} l(d)/\lambda(d)) \cdot W_{\alpha,p}(G),$$

provided $W_{\alpha,p}(G) \neq 0$ and $\coprod_{q\neq p} G_q$ is finite of exponent k where $l_d = |\{g \in \coprod_{q\neq p} G_q \ : \ o(g) = d\}|$, $\lambda(d)$ is the boundary defined as in ([10], (3.8)) and $n = log_2|id(R)|$,
or

$$W_{\alpha,p}(U(RG)) = max(|\coprod_{q\neq p} G_q|, W_{\alpha,p}(G)),$$

provided $W_{\alpha,p}(G) \neq 0$ and $\coprod_{q\neq p} G_q$ is infinite,
or

$$W_{\alpha,p}(U(RG)) = 0,$$

provided $W_{\alpha,p}(G) = 0$.

Proof: Since $id(R)$ is finite, R possesses 2^n idempotents where n is the number of primitive idempotents of R, say $\{e_1, \cdots, e_n\}$ is such a system. Furthermore, owing to a folklore ring-theoretic fact, one may decompose R like this:

$$R = (Re_1) \oplus \cdots \oplus (Re_n) = (Re_1) \times \cdots \times (Re_n)$$

where each Re_i is an indecomposable subring of R; $i \in [1, n]$. Thus, in view of Proposition 2.2 (b), one can write that $RG = (Re_1)G \times \cdots \times (Re_n)G$, whence $U(RG) = U((Re_1)G) \times \cdots \times U((Re_n)G)$. Applying Lemma 2.1, $W_{\alpha,p}(U(RG)) = W_{\alpha,p}(U((Re_1)G)) + \cdots + W_{\alpha,p}(U((Re_n)G))$. It is readily seen that every Re_i is a perfect ring of characteristic p as well; $1 \leq i \leq n$. Moreover, ([2], Theorem 6 - see also [10], Theorem 3.9) applies to calculate all functions $W_{\alpha,p}(U((Re_i)G))$ where $i \in [1, n]$. Thus we obtain the explicit form of $W_{\alpha,p}(U(RG))$ stated above. □

Remark 2.3. *Unfortunately, there is no result of that type for infinite decompositions of R. For example, take $R = \prod_{n=1}^{\infty} F_n / \oplus_{n=1}^{\infty} F_n$ where all F_n are fields. Therefore, the set of idempotents in R is a quotient of boolean algebras: $id(R) = B/J$ where B is the boolean algebra of subsets of the set $\mathbb{N}$ of natural numbers and J is the ideal of finite subsets. Since $|B| = 2^{\aleph_0}$ and $|J| = \aleph_0$, we get that $|id(R)| = 2^{\aleph_0}$. However, $id(R)$ has no atoms (= primitive idempotents), so no ring direct summand of R is indecomposable.*

One source of the problem is that cardinality information is much stronger in the finite case: in fact, any finite boolean algebra is generated by its atoms, so if $|id(R)| = 2^n$, then $id(R)$ is set-theoretically isomorphic to the boolean algebra of subsets of $\{1, \cdots, n\}$ and thus $id(R)$ always possesses primitive idempotents. Consequently, a more promising hypothesis would be to assume that $id(R)$ is isomorphic to the boolean algebra 2^I of subsets of an infinite set I. Nevertheless, it looks like even this is not completely sufficient. For instance, start with $S = \prod_{n=1}^{\infty} F_n$ where each F_n is a copy of some large field F (larger than its prime subfield), choose a nontrivial maximal ideal M in S (meaning one that contains $\oplus_{n=1}^{\infty} F_n$), and take $R = K \cdot 1 + M$, where K is a proper subfield of F. Then R contains all the idempotents of S, so that $id(R) \cong 2^{\mathbb{N}}$, but R is not an infinite direct product of indecomposable rings. E.g., since R is a commutative von Neumann regular ring, it could only be a direct product of indecomposable rings if it were a direct product of fields. That fact would imply R is self-injective, but it is not - in fact, its injective hull, equal to its maximal quotient ring, is S.

We now start the procedure for giving up of a useful algorithm calculating successfully $W_{\alpha,p}(U(RG))$ in a rather general situation for an arbitrary p-divisible group G and with a restriction only on the coefficient ring R. To this aim, suppose R is a ring in which every finitely generated (in particular, each indecomposable) subring is pure – we may also take R to be perfect finitely generated.

And so, let $x \in U(RG)/U^p(RG) = U(RG)/U(R^pG^p) = U(RG)/U(R^pG)$, where the last equality follows by taking into account that $G = G^p$. Thus $x \in U(LG)U(R^pG)/U(R^pG) \cong U(LG)/(U(LG) \cap U(R^pG)) = U(LG)/U((L \cap R^p)G) =$

$U(LG)/U(L^pG)$ for some finitely generated subring L of R containing the same identity as that of R. Furthermore, $L \cong R_1 \times \cdots \times R_n$ where each R_i is indecomposable ($1 \leq i \leq n$), and hence $LG \cong R_1G \times \cdots \times R_nG$ with $U(LG) \cong U(R_1G) \times \cdots \times U(R_nG)$ and $U(L^pG) \cong U(R_1^pG) \times \cdots \times U(R_n^pG)$ under the same isomorphism. We consequently will have $U(LG)/U(L^pG) \cong [U(R_1G)/U(R_1^pG)] \times \cdots \times [U(R_nG)/U(R_n^pG)]$, whence we may formally write $x \in [U(R_1G)/U(R_1^pG)] \times \cdots \times [U(R_nG)/U(R_n^pG)]$. Finally, $U(RG)/U(R^pG) = \cup([U(R_1G)/U(R_1^pG)] \times \cdots \times [U(R_nG)/U(R_n^pG)])$, where the union is taken over each finite family $\{R_i\}_{1\leq i\leq n}$ of indecomposable subrings R_i of R.

On the other hand, if we calculate separately $W_{\alpha,p}(U(R_iG))$ for each index i, then utilizing some set-theoretical gymnastic, there is a way to compute $W_{\alpha,p}(U(RG))$ as well. However, this will be the theme of some other research exploration.

Remark 2.4. *Note also that if G_0/G_p is finite, then $G = M \times K$ where M is finite p-divisible and K is p-mixed. Therefore, $U(RG) \cong U(RM) \times V((RM)K)$ and $V(RG) \cong V(RM) \times V((RM)K)$. Thus, in accordance with Lemma 2.1, the Warfield p-invariants of $U(RG)$ and $V(RG)$ are respectively sums of the Warfield p-invariants of $U(RM)$ plus these of $V((RM)K)$, and of the Warfield p-invariants of $V(RM)$ plus these of $V((RM)K)$. But the Warfield p-invariants of $V((RM)K)$ are completely calculated in [4] because char$(RM) = p$. So, what remains to compute are $W_{\alpha,p}(U(RM))$ or $W_{\alpha,p}(V(RM))$. In this aspect does it follow that $|V(RM)/V(R^pM)| = |R/R^p|$?*

Finally, we assert that if K is a commutative indecomposable unital ring and G is a finite abelian group of exponent which inverts in K, then $KG \cong KH$ for some group H if, and only if, H is finite of the same exponent as that of G and $KG_p \cong KH_p$ for every prime p. The complete proof will be the theme of some other research exploration.

Correction: In [6], pp.7-8 there is a series of identical typos. In fact, in ([6], p. 8, Claim) the equality $|\cup_{i\in I} A_i| = \sum_{i\in I} |A_i|$ should be read and written as $|\cup_{i\in I} A_i| \leq \sum_{i\in I} |A_i|$. In general, an equality cannot be happen. The next two examples manifestly demonstrate this.

If $A_i = A_j$ for all indexes i and j, or $A_i \supseteq A_{i+1}$ for all indices $i \in I$, the equality is trivially false.

A less trivial construction is the following: There exist continuum ($= c$) countable subsets A_i ($i \in c$) of $\mathbb{Z} \oplus \mathbb{Z}$ such that $A_i \cap A_j$ is finite for all $i \neq j$, and $\cup_{i\in c} A_i = \mathbb{Z} \oplus \mathbb{Z}$. Therefore, $|\cup_{i\in c} A_i| = \aleph_0$ while $\sum_{i\in c} |A_i| = c$. The examples are shown.

However, if all sets A_i are disjoint (i.e., $A_i \cap A_j = \emptyset$ for all indices i and j), the desired equality holds, that is, $|\cup_{i\in I} A_i| = \sum_{i\in I} |A_i|$ - see, e.g., Dugundji, Topology, Allyn and Bacon, Boston, 1966, p.30.

So, the statement of Proposition 2.8 on p.7, the equality for $W_{\alpha,p}(U(RG))$ should be written as the inequality "$\leq$". The same correction appears two more times on lines 4 and 8 after the Claim.

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