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Factorizations of the Negatively Subscripted Balancing and Lucas-Balancing Numbers

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abstract: In this paper, we find some tridigonal matrices whose determinant and permanent equal to the negatively subscripted balancing and Lucas-balancing numbers. Also using the first and second kind of Chebyshev polynomials, we obtain factorization of these numbers.

Key Words: Balancing numbers, Lucas - balancing numbers, Triangular numbers, Tridiagonal matrices, Determinant, Permanent.

Contents

1. Introduction

It is well-known that the sequence of balancing numbers $\{B_n\}$ are solutions of the recurrence relation, for $n \geq 2$,

$$B_{n+1} = 6B_n - B_{n-1} \tag{1}$$

with $B_1 = 1, B_2 = 6$ [1]. Also in [1], it is shown that, if x is a balancing number, then $8x^2 + 1$ is a perfect square. If x is balancing number then the positive square root of $8x^2 + 1$ is called a Lucas-balancing number denoted by C_n [8]. Observe that $C_1 = 3, C_2 = 17$ and the Lucas-balancing numbers C_n satisfy the recurrence relation

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2, \tag{2}$$

identical with that for balancing numbers. By using the formulas (1) and (2), we can extend these sequences backward, to get

$$B_{-n} = 6B_{-n+1} - B_{-n+2} = -B_n$$
$$C_{-n} = 6C_{-n+1} - C_{-n+2} = C_n.$$

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In [8], Panda has shown that, the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. In [7], Panda and Ray have proved that the Lucas-balancing numbers are nothing but the even ordered terms of the associated Pell sequence. Also they have shown that the n^{th} balancing numbers are product of n^{th} Pell numbers and n^{th} associated Pell numbers. In [3], K. Liptai searched for those balancing numbers which are Fibonacci numbers too. He proved that the only Fibonacci number in the sequence of balancing numbers is 1. In a similar manner, in [4], he proved that there are no Lucas numbers in the sequence of balancing numbers. L. Szalay in [5] got the same result. In [9], by using Chebyshev polynomials of first and second kind, Ray has obtained nice product formulae for both balancing and Lucas-balancing numbers.

In this paper, we consider negatively subscripted balancing and Lucas-balancing numbers and find some tridiagonal matrices whose determinant and permanent equal to these numbers. In the final section of the paper, we give the factorization of these numbers by using the first and second kinds of Chebyshev polynomials.

2. Negatively Subscripted Balancing and Lucas-balancing Numbers

In this section, we define some tridiagonal matrices and then prove that the determinant and permanent of these matrices are equal to the negatively subscripted balancing and Lucas-balancing numbers. For simplicity, we present some known definitions which will be used subsequently.

Definition 2.1. [2,6]. If $A = (a_{ij})$ is a square matrix of order n, then the permanent of A, denoted by $perA$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{1 \leq i \leq n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n.

There are many applications of permanent that are given in [6].

Definition 2.2. [2]. If $A = (a_{ij})$ is an $m \times n$ matrix with row vectors $r_1, r_2, \ldots r_m$, then A is called contractible on column (resp. row) k if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the matrix $A_{ij:\,k}$ of order $(m-1) \times (n-1)$ can be obtained from A by replacing row i with $a_{jk} r_i + a_{ik} r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:\,ij} = [A^T_{ij:\,k}]^T$ is called the contraction of A on row k relative to columns i and j.

Definition 2.3. [2]. A matrix A is called convertible if there exists an $n \times n$ (1, -1)–matrix H such that $\det(A \circ H) = perA$, where $A \circ H$ is well known Hadamard product of A and H. We call H is a converter of A.

In this paper, every contraction will be on the first column using the first and second rows. It is well known that, if A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then,

$$perA = perB. \tag{3}$$

First, we start with negatively subscripted balancing numbers. We introduce the sequence of matrices $\{M_n, n = 1, 2, \cdots\}$, where M_n is an $n \times n$ tridiagonal matrix with entries $m_{jj} = -6,\ 1 \leq j \leq n$ and $m_{j-1,j} = -1,\ m_{j,j-1} = 1,\ 2 \leq j \leq n$. Then,

$$M_n = \begin{pmatrix} -6 & -1 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix},$$

and the following theorem holds.

Theorem 2.4. *If the sequence of tridiagonal matrices $\{M_n, n = 1, 2, 3 \cdots\}$ is of the form*

$$M_n = \begin{pmatrix} -6 & -1 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix},$$

then

$$perM_n = (-1)^{n-1} B_{-(n+1)},$$

where B_{-n} is the n^{th} negatively subscripted balancing number.

Proof: Clearly for $n = 1, n = 2$, we have

$$\begin{aligned} perM_1 &= -6 = B_{-2} \\ perM_2 &= 35 = -B_{-3}. \end{aligned}$$

Let the p^{th} contraction of M_n be M_n^p, where $1 \leq p \leq n-2$. Using Definition 2.2, the matrix M_n can be contracted on Column 1 as

$$M_n^1 = \begin{pmatrix} 35 & 6 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -B_{-3} & -B_{-2} & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}.$$

Again the matrix M_n^1 can be contracted on Column 1 as,

$$M_n^2 = \begin{pmatrix} -204 & -35 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -B_{-4} & -B_{-3} & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}.$$

Proceeding in this way, we obtain, for $3 \le r \le n-4$,

$$M_n^r = \begin{pmatrix} (-1)^r B_{-(r+2)} & (-1)^r B_{-(r+1)} & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix},$$

and therefore

$$M_n^{n-3} = \begin{pmatrix} (-1)^{n-3} B_{-(n-1)} & (-1)^{n-3} B_{-(n-2)} & 0 \\ 1 & -6 & -1 \\ 0 & 1 & -6 \end{pmatrix}.$$

By contraction of M_n^{n-3} on Column 1, gives

$$M_n^{n-2} = \begin{pmatrix} (-1)^{n-3}(-6)B_{-(n-1)} + (-1)^{n-3}B_{-(n-2)} & (-1)^{n-2}B_{-(n-1)} \\ 1 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{n-2}[6B_{-(n-1)} - B_{-(n-2)}] & (-1)^{n-2}B_{-(n-1)} \\ 1 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{n-2}B_{-n} & (-1)^{n-2}B_{-(n-1)} \\ 1 & -6 \end{pmatrix}.$$

Using (3), we get

$$\begin{aligned} perM_n &= perM_n^{n-2} \\ &= (-1)^{n-2}(-6)B_{-n} + (-1)^{n-2}B_{-(n-1)} \\ &= (-1)^{n-1}\left[6B_{-n} - B_{-(n-1)}\right] \\ &= (-1)^{n-1}B_{-(n+1)}, \end{aligned}$$

which ends the proof of the theorem. □

For negatively subscripted Lucas-balancing numbers, consider another $n \times n$ tridiagonal matrix $D_n = (d_{ij})$, with $d_{ii} = -6$ for $2 \leq i \leq n, d_{i,i+1} = -1, d_{i+1,i} = 1$ for $1 \leq i \leq n-1, d_{11} = -3$, then

$$D_n = \begin{pmatrix} -3 & -1 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}.$$

We prove the following theorem.

Theorem 2.5. *If the sequence of tridiagonal matrices $\{D_n, n = 1, 2, 3 \cdots\}$ is of the form*

$$D_n = \begin{pmatrix} -3 & -1 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix},$$

then

$$perD_n = (-1)^{n-2}C_{-n},$$

where C_{-n} is the n^{th} negatively subscripted Lucas-balancing number.

Proof: The theorem holds for $n = 1, n = 2$, because

$$perD_1 = -3 = -C_{-1},$$
$$perD_2 = 17 = C_{-2}.$$

Let D_n^p be the p^{th} contraction of D_n where $1 \leq p \leq n-2$. By virtue of Definition 2.2, the matrix D_n can be contracted on Column 1 as

$$D_n^1 = \begin{pmatrix} 17 & 3 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}$$
$$= \begin{pmatrix} C_{-2} & C_{-1} & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}.$$

The matrix D_n^1 can be contracted on Column 1 as,

$$D_n^2 = \begin{pmatrix} -99 & -17 & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}$$
$$= \begin{pmatrix} -C_{-3} & -C_{-2} & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix}.$$

Continuing in this way, we obtain, for $3 \leq r \leq n-4$,

$$D_n^r = \begin{pmatrix} (-1)^{r-1}C_{-(r+1)} & (-1)^{r-1}C_{-r} & & & \\ 1 & -6 & -1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -6 \end{pmatrix},$$

and therefore

$$D_n^{n-3} = \begin{pmatrix} (-1)^{n-4}C_{-(n-2)} & (-1)^{n-4}C_{-(n-3)} & 0 \\ 1 & -6 & -1 \\ 0 & 1 & -6 \end{pmatrix}.$$

By contraction of D_n^{n-3} on Column 1, gives

$$D_n^{n-2} = \begin{pmatrix} (-1)^{n-4}(-6)C_{-(n-2)} + (-1)^{n-4}C_{-(n-3)} & (-1)^{n-3}C_{-(n-2)} \\ 1 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{n-3}[6C_{-(n-2)} - C_{-(n-3)}] & (-1)^{n-3}C_{-(n-2)} \\ 1 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{n-3}C_{-(n-1)} & (-1)^{n-3}C_{-(n-2)} \\ 1 & -6 \end{pmatrix}.$$

Again by using (3), we get

$$\begin{aligned} perD_n &= perD_n^{n-2} \\ &= (-1)^{n-3}(-6)C_{-(n-1)} + (-1)^{n-3}C_{-(n-2)} \\ &= (-1)^{n-2}\left[6C_{-(n-1)} - C_{-(n-2)}\right] \\ &= (-1)^{n-2}C_{-n}. \end{aligned}$$

This completes the proof of the theorem. □

By virtue of Definition 2.3, we need a suitable matrix H for Hadamard product. Since H is an $n \times n$ (-1, 1)–matrix, we can write

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{pmatrix}.$$

Let the Hadamard products $M_n \circ H$ and $D_n \circ H$ respectively denoted by the matrices P_n and Q_n be given by

$$P_n = \begin{pmatrix} -6 & -1 & & & \\ -1 & -6 & -1 & & \\ & -1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -6 \end{pmatrix}$$

$$Q_n = \begin{pmatrix} -3 & -1 & & & \\ -1 & -6 & -1 & & \\ & -1 & -6 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -6 \end{pmatrix}.$$

It is well known that the value of the following determinant,

$$\det \begin{pmatrix} a & x & & \\ \frac{1}{x} & a & \ddots & \\ & \ddots & \ddots & x \\ & & \frac{1}{x} & a \end{pmatrix}$$

is independent of x (see p.105, [10]).

Therefore, using the above result and considering the following matrices

$$\hat{P}_n = \begin{pmatrix} -6 & 1 & & & \\ 1 & -6 & 1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -6 \end{pmatrix}$$

$$\hat{Q}_n = \begin{pmatrix} -3 & 1 & & & \\ 1 & -6 & 1 & & \\ & 1 & -6 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -6 \end{pmatrix},$$

we can write

$$\det(\hat{P}_n) = \det P_n = perM_n = (-1)^{n-1} B_{-(n+1)}, \tag{4}$$

$$\det(\hat{Q}_n) = \det Q_n = perD_n = (-1)^{n-2}C_{-n}. \tag{5}$$

3. Factorization of Negatively Subscripted Balancing and Lucas-balancing Numbers

In this section we find the eigenvalues of the two tridiagonal matrices whose determinants are associated with the negatively subscripted balancing and Lucas-balancing numbers. Then with the help of Chebyshev polynomials of first and second kind, we also obtain the factorization of these numbers.

Theorem 3.1. *If B_{-n} is the n^{th} negatively subscripted balancing number, then for $n \geq 1$*

$$B_{-(n+1)} = (-1)^{n-1} \prod_{k=1}^{n} \left[-6 - 2\cos\left(\frac{\pi k}{n+1}\right)\right].$$

Proof: We introduce another $n \times n$ tridiagonal matrix $R_n = (b_{ij})$ with $b_{ii} = 0$ for $1 \leq i \leq n$ and $b_{i,i-1} = b_{i-1,i} = 1$ for $2 \leq i \leq n$. Then,

$$R_n = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

We observe that $\hat{P}_n = -6I + R_n$. Let $\alpha_k, 1 \leq k \leq n$ be the eigenvalues of R_n with corresponding eigenvectors X_k. Then for all k

$$\begin{aligned} \hat{P}_n X_k &= (-6I + R_n)X_k \\ &= -6IX_k + R_n X_k \\ &= -6X_k + \alpha_k X_k \\ &= (-6 + \alpha_k)X_k. \end{aligned}$$

This shows that $-6 + \alpha_k,\ 1 \leq k \leq n$, are eigenvalues of $\hat{P}_n$. Thus for $n \geq 1$, we have

$$\det \hat{P}_n = \prod_{k=1}^{n} [-6 + \alpha_k]. \tag{6}$$

Recall that each α_k is a root of the characteristic polynomial $p(\alpha) = \det(R_n - \alpha I)$, and since $R_n - \alpha I$ is a tridiagonal matrix, that is,

$$R_n - \alpha I = \begin{pmatrix} -\alpha & 1 & & & \\ 1 & -\alpha & 1 & & \\ & 1 & -\alpha & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\alpha \end{pmatrix},$$

we get the following recursive formulas for the characteristic polynomials:

$$\begin{aligned} p_1(\alpha) &= -\alpha \\ p_2(\alpha) &= \alpha^2 - 1 \\ p_n(\alpha) &= -\alpha p_{n-1}(\alpha) - p_{n-2}(\alpha). \end{aligned}$$

This family of polynomials can be transformed into another family $\{U_n(x), n \geq 1\}$ by the transformation $\alpha = -2x$ to get,

$$\begin{aligned} U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x). \end{aligned}$$

Observe that the family $\{U_n(x), n \geq 1\}$ is a set of Chebyshev polynomials of second kind. It is well known that for $x = \cos\theta$, the Chebyshev polynomials of the second kind can be written as

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta},$$

which when equal to zero gives $\theta_k = \frac{\pi k}{n+1},\ k = 1, 2, \cdots, n$. Therefore, we get

$$\begin{aligned} x_k &= \cos\theta_k \\ &= \cos\frac{\pi k}{n+1},\ k = 1, 2, \cdots, n. \end{aligned}$$

The transformation $\alpha = -2x$, gives the eigenvalues of R_n as

$$\alpha_k = -2\cos\frac{\pi k}{n+1}, k = 1, 2, \cdots, n. \tag{7}$$

Thus from (4), (6) and (7), we get the desired result as

$$B_{-(n+1)} = \det \hat{P}_n = (-1)^{n-1} \prod_{k=1}^{n} \left[-6 - 2\cos\left(\frac{\pi k}{n+1}\right)\right].$$

□

Theorem 3.2. *If C_{-n} is the n^{th} negatively subscripted Lucas-balancing number, then for $n \geq 1$*

$$C_{-n} = \frac{(-1)^{n-2}}{2} \prod_{k=1}^{n} \left[-6 - 2\cos\left(\frac{\pi(2k-1)}{2n}\right)\right].$$

Proof: From equation (5), we have $\det \hat{Q}_n = (-1)^{n-2} C_{-n}$. If e_j is the j^{th} column of the identity matrix I, we observe that $det(I + e_1 e_1^T) = 2$. Thus we may write

$$\det \hat{Q} = \frac{1}{2} \det \left[(I + e_1 e_1^T)\hat{Q}\right]. \tag{8}$$

Also observe that the right hand side of (8) can be expressed as

$$\frac{1}{2}\det\left[(I+e_1e_1^T)\hat{Q}\right]=\frac{1}{2}\det\left[-6I+R_n+e_1e_2^T\right].$$

If $\gamma_k, k=1,2,\cdots,n$, are the eigenvalues of $R_n+e_1e_2^T$ with corresponding eigenvectors Y_k, then for each j,

$$\begin{aligned}[-6I+R_n+e_1e_2^T]Y_j &= -6IY_j+(R_n+e_1e_2^T)Y_j\\ &= -6Y_j+\gamma_jY_j\\ &= (-6+\gamma_j)Y_j.\end{aligned}$$

This shows that the eigenvalues of the matrix $\left(-6I+R_n+e_1e_2^T\right)$ are $-6+\gamma_k$, where $1\le k\le n$. Therefore,

$$\det\hat{Q}=\frac{1}{2}\det\left[-6I+R_n-ie_1e_2^T\right]=\frac{1}{2}\prod_{1\le k\le n}(-6+\gamma_k),\quad n\ge 1. \tag{9}$$

In order to find $\gamma_k's$, we recall that each γ_k is a zero of the characteristic polynomial $q_n(\gamma)=det(R_n+e_1e_2^T-\gamma I)$. Since $\det\left(I-\frac{1}{2}e_1e_1^T\right)=\frac{1}{2}$, we can express the characteristic polynomial as

$$\begin{aligned}q_n(\gamma)&=2\det\left[(I-\frac{1}{2}e_1e_1^T(R_n+e_1e_2^T-\gamma I)\right]\\ &=2\det\begin{pmatrix}-\frac{\gamma}{2} & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & \ddots & \\ & & \ddots & \ddots & 1\\ & & & 1 & -\gamma\end{pmatrix}.\end{aligned}$$

We obtain the recursive formulas:

$$\begin{aligned}q_1(\gamma)&=-\frac{\gamma}{2},\\ q_2(\gamma)&=\frac{\gamma^2}{2}-1,\\ q_n(\gamma)&=-\gamma q_{n-1}(\gamma)-q_{n-2}(\gamma).\end{aligned}$$

Using the transformation $\gamma=-2x$, the family of the above polynomial can be transformed to a new family $\{T_n(x), n\ge 1\}$ where,

$$\begin{aligned}T_1(x)&=x\\ T_2(x)&=2x^2-1\\ T_n(x)&=2xT_{n-1}(x)-T_{n-2}(x).\end{aligned}$$

Again observe that the family $\{T_n(x), n \geq 1\}$ is a set of Chebyshev polynomials of first kind. It is well known that, for $x = \cos\theta$, the Chebyshev polynomials of the first kind can be written as $T_n(x) = \cos n\theta$, which when equal to zero gives $\theta_k = \frac{\pi(2k-1)}{2n}, k = 1, 2, \cdots, n$. Thus

$$\begin{aligned} x_k &= \cos\theta_k \\ &= \cos\frac{\pi(2k-1)}{2n}, k = 1, 2, \cdots, n. \end{aligned}$$

Applying the transformation $\gamma = -2x$, the eigenvalues of $R_n + e_1 e_2^T$ are given by

$$\gamma_k = -2\cos\frac{\pi(2k-1)}{2n}, k = 1, 2, \cdots, n. \tag{10}$$

Thus from equations (8), (9) and (10), we get

$$C_{-n} = \frac{(-1)^{n-2}}{2}\prod_{k=1}^{n}\left[-6 - 2\cos\left(\frac{\pi(2k-1)}{2n}\right)\right],$$

which ends the proof of the theorem. □

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