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Influence of weakly $\mathcal{H}$-subgroups of minimal subgroups on the structure of finite groups *

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abstract: Let G be a finite group. A subgroup H of G is called an $\mathcal{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$ for all $g \in G$. A subgroup H of G is called a weakly $\mathcal{H}$-subgroup in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is an $\mathcal{H}$-subgroup in G. In this paper, we use weakly $\mathcal{H}$-subgroup condition on minimal subgroups to study the structure of the finite group G. Some earlier results are improved and extend.

Key Words: c-normal subgroup; $\mathcal{H}$-subgroup; weakly $\mathcal{H}$-subgroup; p-nilpotent group; Fitting subgroup; saturated formation.

Contents

1. Introduction

Throughout this paper, all groups are finite. Our notation is standard and taken mainly from Doerk and Hawkes [8].

A question of particular interest in the theory of groups is to study the structure of a group G by using a certain generalized normality of some subgroups of G. Wang [16] introduced the concept of c-normality of a subgroup of a finite group as follows: A subgroup H of a group G is said to be c-normal in G if G has a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = \cap_{g \in G} H^g$ is the core of H in G, that is, the largest normal subgroup of G contained in H. Bianchi et al. [5] introduced the following concept: A subgroup H of a group G is said to be an $\mathcal{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$ for all $g \in G$. In [2], Asaad, Heliel and Al-Shomrani introduced a new subgroup embedding property of a finite group, called a weakly $\mathcal{H}$-subgroup, which is a generalization of both c-normality and $\mathcal{H}$-subgroup. A subgroup H of a group G is said to be a weakly $\mathcal{H}$-subgroup in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \in \mathcal{H}(G)$, where $\mathcal{H}(G)$ denotes the set of all $\mathcal{H}$-subgroups of a group G. It is clear that each

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of c-normality and $\mathcal{H}$-subgroup concepts imply weakly $\mathcal{H}$-subgroup. The converse does not hold in general. The following two examples appear in Asaad et al. [2]:

Example 1.1. *Set $H = S_5$, the symmetric group of degree 5, and $L = A_5$, the alternating group of degree 5. Let K be a group of order 2 and consider $G = H \times K$. Let P be a Sylow 2- subgroup of H and P_1 be a Sylow 2-subgroup of L . Clearly, P is not c-normal in G as $H = S_5$ is a non solvable group. On the other hand, $G = P(KL)$ and $KL \trianglelefteq G$. Also, $P \cap KL = P \cap KP_1 = P_1(P \cap K) = P_1$. Now, consider $N_G(P_1) \cap P_1^g$, $g \in G$. Clearly, $g \in G = H \times K$ implies that $N_G(P_1) \cap P_1^x \leq P^y \cap P_1^x \leq P^y \cap L^x = P^y \cap L = P_1$ for some $x \in H$, $y \in N_G(P_1)$ (note that $P_1 \leq P^y$). Thus $P \cap KL = P_1 \in \mathcal{H}(G)$. Hence P is weakly $\mathcal{H}$-subgroup in G.*

Example 1.2. *Let H be a cyclic group of order 5 and let K be the full automorphism group of H of order 4. Set $G = [H]K$ and let L be a subgroup of K of order 2. Clearly, L is not c-normal in G. On the other hand, $G = LG$ and, since $N_G(L) = K$, we have that $N_G(L) \cap L^x = K \cap L^x$, $x \in H$. Hence, $N_G(L) \cap L^x \leq L^x$ or $N_G(L) \cap L^x = 1$. In all cases, $N_G(L) \cap L^x \leq L$ which means that $L \in \mathcal{H}(G)$. Thus L is weakly $\mathcal{H}$-subgroup in G.*

A minimal subgroup of a group G is a subgroup of prime order. How minimal subgroups can be embedded in a group G is a question of particular interest in studying the structure of G. In fact, many authors have investigated the influence of normality, c-normality, $\mathcal{H}$-subgroup and more recently weakly $\mathcal{H}$-subgroup of the minimal subgroups of a group G on the structure of G; see for example [1], [3,4], [6,7], [10], [12,20] and [22,23]. The present paper may be viewed as a continuation of [1], [17] and [20]. More precisely, we study the influence of weakly $\mathcal{H}$-subgroups of the minimal subgroups of a group G on its structure. Some earlier results are improved and extend.

2. Basic definitions and preliminaries

In this section, for the sake of convenience, we collect some definitions and state some known results from the literature which will be used in the sequel.

Recall that a class of groups $\mathfrak{F}$ is a formation provided that the following conditions are satisfied:

(1) If $G \in \mathfrak{F}$, then $G/N \in \mathfrak{F}$, where N is any normal subgroup of G.

(2) If G/M and G/N are both in $\mathfrak{F}$, then $G/(M \cap N)$ is also in $\mathfrak{F}$ for any normal subgroups M and N of G.

A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that G belongs to $\mathfrak{F}$. Throughout this paper, $\mathfrak{U}$ and $\mathfrak{N}$ will denote the classes of supersolvable groups and nilpotent groups, respectively. It is known that $\mathfrak{U}$ and $\mathfrak{N}$ are saturated formations.

A normal subgroup N of a group G is an $\mathfrak{F}$-hypercentral subgroup of G provided N possesses a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq ... \trianglelefteq N_s = N$ such that N_{i+1}/N_i is an $\mathfrak{F}$-central chief factor of G (see [8], p. 387). The product of all $\mathfrak{F}$-hypercentral subgroups of G is again an $\mathfrak{F}$-hypercentral subgroup, denoted by $Z_{\mathfrak{F}}(G)$, and called the $\mathfrak{F}$-hypercenter of G (see [8], IV 6.8). For the formation $\mathfrak{N}$ of nilpotent groups, the $\mathfrak{N}$-hypercenter of a group G is simply the terminal member $Z_\infty(G)$ of the ascending central series of G.

Lemma 2.1. *Let L be a subgroup of a group G.*

(i) *If L is a weakly $\mathcal{H}$-subgroup in G, $L \leq H \leq G$, then L is a weakly $\mathcal{H}$-subgroup in H.*

(ii) *Let $N \trianglelefteq G$ and $N \leq L$. Then L is a weakly $\mathcal{H}$-subgroup in G if and only if L/N is a weakly $\mathcal{H}$-subgroup in G/N.*

(iii) *Let L be a p-subgroup of G and N a normal p'-subgroup of G. If L is a weakly $\mathcal{H}$-subgroup in G, then LN/N is a weakly $\mathcal{H}$-subgroup in G/N.*

Proof: See [2], Lemma 2.2. □

Lemma 2.2. *Let G be a minimal non p-nilpotent group (A non p-nilpotent group all of whose proper subgroups are p-nilpotent), where p is a prime. Then*

(i) *G is a minimal non-nilpotent group (A non-nilpotent group all of whose proper subgroups are nilpotent).*

(ii) *$G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G.*

(iii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iv) *If $p > 2$, then the exponent of P is p and, when $p = 2$, the exponent of P is at most 4.*

Proof: See [11], Satz 5.4, p. 434; and Satz 5.2, p. 281. □

Lemma 2.3. *Suppose that G is a group and P is a normal p-subgroup of G contained in $Z_\infty(G)$. Then $O^p(G) \leq C_G(P)$.*

Proof: See [21], Lemma 2.8. □

Lemma 2.4. *Let G be a group and let $H \in \mathcal{H}(G)$.* If $H \trianglelefteq\trianglelefteq K \leq G$, then $H \trianglelefteq K$.

Proof: See [5], Theorem 6(2). □

Lemma 2.5.*Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups$\mathfrak{U}$. A group G lies in $\mathfrak{F}$ if and only if it has a solvable normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order or of order 4 is a weakly $\mathcal{H}$-subgroup in G.*

Proof: See [1], Corollary 3.12. □

Lemma 2.6. *If A is a $\acute{p}$-group of automorphisms of the abelian p-group P which acts trivially on $\Omega_1(P)$, then $A = 1$.*

Proof: See [9], Theorem 2.4, p. 178. □

3. Results

We first prove the following result:

Theorem 3.1. *Assume that G is a group, p is a fixed prime number and every subgroup of order p is contained in $Z_\infty(G)$. In addition, if $p = 2$, assume that every cyclic subgroup of order 4 of G is a weakly $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$, then G is p-nilpotent.*

Proof: Let G be a counterexample of minimal order. Then we have:

(1) G is a minimal non p-nilpotent group.

Let H be any proper subgroup of G, and let L be a cyclic subgroup of H of order p (or of order 4 if $p = 2$). Then $L \leq Z_\infty(G) \cap H \leq Z_\infty(H)$. By Lemma 2.1(i), if L is a weakly $\mathcal{H}$-subgroup in G, then L is a weakly $\mathcal{H}$-subgroup in H. Thus H satisfies the hypothesis of the theorem in any case. The minimal choice of G implies that H is p-nilpotent. Thus G is a minimal non p-nilpotent group. From Lemma 2.2, G is a minimal non-nilpotent group, $G = PQ$, $P \trianglelefteq G$, $Q \ntrianglelefteq G$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(2) $p = 2$ and every element of order 4 is a weakly $\mathcal{H}$-subgroup in G.

If not, $p > 2$. By Lemma 2.2(iv), the exponent of P is p. Thus, by hypothesis, $P \leq Z_\infty(G)$. By Lemma 2.3, $O^p(G) \leq C_G(P)$ and so $G = PQ = P \times Q$ is nilpotent, a contradiction. If every element of order 4 of G lies in $Z_\infty(G)$, then again $P \leq Z_\infty(G)$ and we have a contradiction.

(3) For every element $x \in P - \Phi(P)$, $|x| = 4$.

If not, there exists $x \in P - \Phi(P)$ such that $|x| = 2$. Clearly, $< x^G > \leq P$ as $P \trianglelefteq G$. Then $< x^G > \Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup in $G/\Phi(P)$ from (1), we have that $P =< x^G > \Phi(P) =< x^G > \leq Z_\infty(G)$ and so G is nilpotent, a contradiction.

(4) Final contradiction.

By (3), for every element $x \in P - \Phi(P)$, $|x| = 4$. By (2), $< x >$ is a weakly $\mathcal{H}$-subgroup in G. Then there exists a normal subgroup K of G such that $G =< x > K$ and $< x > \cap K \in \mathcal{H}(G)$. Clearly, $P \cap K \trianglelefteq G$ and so $(P \cap K)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Hence $P \cap K \leq \Phi(P)$ or $P \cap K = P$ as $P/\Phi(P)$ is a minimal normal subgroup in $G/\Phi(P)$ from (1). If $P \cap K \leq \Phi(P)$, then $P = P \cap G = P \cap < x > K =< x > (P \cap K) =< x > \Phi(P)$. Hence, $P =< x >$ is a cyclic normal Sylow 2-subgroup of G of order 4 which means that G is nilpotent, a contradiction. Therefore, we may assume that $P \cap K = P$. Then, $< x >=< x > \cap P =< x > \cap (P \cap K) =< x > \cap K \in \mathcal{H}(G)$. Since $< x > \trianglelefteq\trianglelefteq G$, we have, by Lemma 2.4, that $< x > \trianglelefteq G$ and hence $< x > Q < G$ (if $< x > Q = G$, we have a contradiction as we mentioned).

By (1), $< x > Q =< x > \times Q$ and hence $< x > \leq N_G(Q)$. Thus, $P \leq N_G(Q)$, a final contradiction completing the proof of the theorem. □

Since a group G is nilpotent if and only if it is p-nilpotent for every prime p dividing the order of G, the following corollaries are immediate consequences of Theorem 3.1:

Corollary 3.2. *Let G be a group. If every minimal subgroup of G is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of G is a weakly $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 3.3. *Let G be a group. If every minimal subgroup of G is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of G is an $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 3.4. *Let G be a group. If every minimal subgroup of G is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of G is c-normal in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Now we can prove:

Theorem 3.5. *Assume that N is a normal subgroup of a group G such that G/N is p-nilpotent, where p is a fixed prime number, and every subgroup of order p of N is contained in $Z_\infty(G)$. In addition, if $p = 2$, assume that every cyclic subgroup of order 4 of N is a weakly $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$, then G is p-nilpotent.*

Proof: Let G be a counterexample of minimal order. Then we have:

(1) G is a minimal non p-nilpotent group.

Let H be any proper subgroup of G. Since G/N is p-nilpotent, we have that $H/(H \cap N) \cong HN/N$ is p-nilpotent as the class of p-nilpotent groups is a subgroup closed. Let L be a cyclic subgroup of H of order p (or of order 4 if $p = 2$). Then $L \leq Z_\infty(G) \cap H \leq Z_\infty(H)$. By Lemma 2.1(i), if L is a cyclic subgroup of order 4 of $H \cap N$ which is a weakly $\mathcal{H}$-subgroup in G, then L is a weakly $\mathcal{H}$-subgroup in H. Thus H, $H \cap N$ satisfy the hypothesis of the theorem in any case. The minimal choice of G implies that H is p-nilpotent. Thus G is a minimal non p-nilpotent group. From Lemma 2.2, G is a minimal non-nilpotent group, $G = PQ$, $P \trianglelefteq G$, $Q \ntrianglelefteq G$.

(2) $P \leq N$.

Clearly, from (1), G/P is nilpotent (in particular p-nilpotent) and, since G/N is p-nilpotent, we have that $G/(P \cap N)$ is p-nilpotent. Now if $P \nleq N$, then $P \cap N < P$ and so $Q(P \cap N) < QP = G$. Thus, from (1), $Q(P \cap N)$ is nilpotent and hence $Q(P \cap N) = Q \times (P \cap N)$. Since $G/(P \cap N) = (P/(P \cap N))(Q(P \cap N)/(P \cap N))$ is p-nilpotent, we have that $Q(P \cap N)/(P \cap N) \trianglelefteq G/(P \cap N)$ and so $Q(P \cap N) \trianglelefteq G$. Now Q is characteristic in $Q(P \cap N) \trianglelefteq G$ implies that $Q \trianglelefteq G$, a contradiction. Thus $P \leq N$.

(3) Final contradiction.

If $p > 2$, then, by Lemma 2.2(iv), the exponent of P is p. Thus $P = P \cap N \leq Z_\infty(G)$. By Lemma 2.3, $O^p(G) \leq C_G(P)$ and so $G = PQ = P \times Q$, a contradiction. Assume that $p = 2$. Clearly, as $P \trianglelefteq G$, every element of order 2 or of order 4 of G is contained in P (in particular in N by (2)). Hence every element of order 2 of G lies in $Z_\infty(G)$ and every cyclic subgroup of order 4 of G is a weakly $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$ by hypothesis. Applying Theorem 3.1 yields G is 2-nilpotent, a contradiction completing the proof of the theorem. □

The following corollaries are immediate consequences of Theorem 3.5:

Corollary 3.6. *Let G be a group and N be a normal subgroup of G such that G/N is nilpotent. If every minimal subgroup of N is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of N is a weakly $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 3.7. *Let G be a group and N be a normal subgroup of G such that G/N is nilpotent. If every minimal subgroup of N is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of N is an $\mathcal{H}$-subgroup in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 3.8. *Let G be a group and N be a normal subgroup of G such that G/N is nilpotent. If every minimal subgroup of N is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of N is c-normal in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Corollary 3.9. [17], Theorem 2.4 *Let G be a group and $K = G^{\mathcal{N}}$ be the nilpotent residual of G. Then G is nilpotent if and only if every minimal subgroup of K is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of K is c-normal in G or lies in $Z_\infty(G)$.*

Corollary 3.10. *Let G be a group and assume that every cyclic subgroup of order 4 of G' is a weakly $\mathcal{H}$-subgroup (an* $\mathcal{H}$-subgroup *or c-normal) in G. Then G is nilpotent if and only if every minimal subgroup of G' lies in $Z_\infty(G)$.*

Clearly, Theorem 3.5 and its corollaries already supersede some earlier results such as Yokoyama results [22], Theorems 2.4 and 2.5, [23], Theorem 4, Ballester-Bolinches and Wang [4], Corollary 3.3 and a result of Issacs [12], Lemma B.

The motivation for the next result is as follows: Al-Shomrani, Ramadan and Heliel [1] showed that if G is a group and $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$, then G lies in $\mathfrak{F}$ if it has a solvable normal subgroup H such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order or of order 4 is a weakly $\mathcal{H}$-subgroup in G. This result is not true in general if the assumption "every subgroup of $F(H)$ of prime order or of order 4 is a weakly $\mathcal{H}$-

subgroup in G " is replaced by "every subgroup of prime order of $F(H)$ is a weakly $\mathcal{H}$-subgroup in G". The following example illustrates that:

Example 3.11. Let $\mathfrak{F} = \mathfrak{NU}$ be the class of groups G whose commutator subgroup G' is nilpotent. Clearly, $\mathfrak{F}$ is a formation. If $G/\Phi(G) \in\mathfrak{F}$, then $(G/\Phi(G))' = G'\Phi(G)/\Phi(G)$ is nilpotent and therefore $G'\Phi(G)$ is nilpotent. Then $G' \leq G'\Phi(G)$ implies that G' is nilpotent. Thus $\mathfrak{F}$ is saturated. Also the class $\mathfrak{F}$ contains the class $\mathfrak{U}$ as the commutator subgroup of a supersolvable group is nilpotent. Now consider the group $G = GL(2,3)$. This group has a normal subgroup H isomorphic to the quaternion group of order 8 such that $G/H \cong S_3$, the symmetric group of order 6, and therefore we have $G/H \in\mathfrak{F}$. Notice that the unique subgroup of $F(H)$ with prime order is the center of H and this is not only a weakly $\mathcal{H}$-subgroup in G but also normal in G. Since the commutator subgroup of $G = GL(2,3)$ is $G' = SL(2,3)$, then $G \notin\mathfrak{F}$.

In view of example 3.11, we prove the following theorem:

Theorem 3.12. *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$ and let G be a group with a normal solvable subgroup H such that $G/H \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ under either of the following:*

(i) *G is 2-nilpotent and every subgroup of odd prime of $F(H)$ is a weakly $\mathcal{H}$-subgroup in G.*

(ii) *The Sylow 2-subgroups of G are abelian and every subgroup of prime order of $F(H)$ is a weakly $\mathcal{H}$-subgroup in G.*

Proof. (i) Let G be a counterexample of minimal order. Since G is 2-nilpotent, it follows that H is also 2-nilpotent and therefore $H = PK$, where P is a Sylow 2-subgroup of H and K is a normal Hall $2'$-subgroup of H. As K is characteristic in H and $H \trianglelefteq G$, we have $K \trianglelefteq G$. Clearly, $(G/K)/(H/K) \cong G/H \in \mathfrak{F}$, $F(H/K)$ is 2-group and G/K is 2-nilpotent. Thus G/K satisfies the hypothesis of the theorem. By minimality of G, $G/K \in \mathfrak{F}$. But $F(K) \leq F(H)$, then applying Lemma 2.5 yields $G \in \mathfrak{F}$, a contradiction completing the proof of (i).

(ii) Let G be a counterexample of minimal order and let P be an abelian Sylow 2-subgroup of G. For every element $x \in \Omega_1(P \cap F(H))$, consider the minimal 2-subgroup P^* of G contained in $L =< x^g : g \in G >$ which is the minimal normal subgroup of G containing x. Then $P^* \leq F(H)$. By hypothesis, P^* is a weakly $\mathcal{H}$-subgroup in G. Then there exists a normal subgroup K of G such that $G = P^*K$ and $P^*\cap K \in \mathcal{H}(G)$. Clearly, $G = LK$ and $L\cap K = 1$or $L\cap K = L$ as L is a minimal normal subgroup of G. If $L \cap K = 1$, then $L = L \cap (P^*K) = P^*(L \cap K) = P^*$. In other words, $P^* =< x^g >$ for some $g \in G$. Hence x^g lies in the center of G and so does x. This means that $\Omega_1(P \cap F(H))$ lies in the center of G. By Lemma 2.6, every element of odd order centralizes $P\cap F(H)$ and so $P\cap F(H)$ lies in the center of G as P is abelian. Now, clearly, $G/(P \cap F(H))$ satisfies the hypothesis of the theorem. The minimal choice of G implies that $G/P\cap F(H) \in \mathfrak{F}$. Applying Lemma 2.5 yields $G \in \mathfrak{F}$, a contradiction. Thus we may assume that $L \cap K = L$. Then

$L \leq K$ and so $P^* = P^* \cap K \in \mathcal{H}(G)$. By Lemma 2.4, $P^* \trianglelefteq G$ and hence $P^* = L$ which yields $G \in \mathfrak{F}$ by the above arguments, a final contradiction completing the proof of (ii).

The following corollaries are immediate consequences of Theorem 3.12:

Corollary 3.13. [20], Theorem 4*Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$ and let G be a group with a normal solvable subgroup H such that $G/H \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ under either of the following:.*

(i) *G is 2-nilpotent and every subgroup of odd prime of $F(H)$ is c-normal in G.*

(ii) *The Sylow 2-subgroups of G are abelian and every subgroup of prime order of $F(H)$ is c-normal in G.*

Corollary 3.14. *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$ and let G be a group with a normal solvable subgroup H such that $G/H \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ under either of the following:.*

(i) *G is 2-nilpotent and every subgroup of odd prime of $F(H)$ is an $\mathcal{H}$-subgroup in G.*

(ii) *The Sylow 2-subgroups of G are abelian and every subgroup of prime order of $F(H)$ is an $\mathcal{H}$-subgroup in G.*

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