Bol. Soc. Paran. Mat. (3s.) **v. 31** 2 (2013): 121–128.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i2.16296

One Parameter Family of $\mathfrak{b}-\mathbf{m}_1$ Developable Surfaces of Biharmonic New Type $\mathfrak{b}-$Slant Helices according to Bishop Frame in the Sol Space Sol^3

Mahmut Ergüt, Talat Körpınar and Essin Turhan

ABSTRACT: In this paper, we study inextensible flows of $\mathfrak{b}-\mathbf{m}_1$ developable surfaces of biharmonic new type $\mathfrak{b}-$slant helix in the Sol^3. We characterize one parameter family of the $\mathfrak{b}-\mathbf{m}_1$ developable surfaces in terms of their Bishop curvatures.

Key Words: new type $\mathfrak{b}-$slant helix, Sol space, curvatures, $\mathfrak{b}-\mathbf{m}_1$ developable surface

Contents

1. Introduction

Developable surfaces are defined as the surfaces on which the Gaussian curvature is 0 everywhere. The developable surfaces are useful since they can be made out of sheet metal or paper by rolling a flat sheet of material without stretching it. Most large-scale objects such as airplanes or ships are constructed using un-stretched sheet metals, since sheet metals are easy to model and they have good stability and vibration properties. Moreover, sheet metals provide good fluid dynamic properties. In ship or airplane design, the problems usually stem from engineering concerns and in engineering design there has been a strong interest in developable surfaces [2,3,13].

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ [4-11].

2000 *Mathematics Subject Classification:* 53A04, 53A10

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \operatorname{tr} R\,(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study inextensible flows of $\mathfrak{b}-\mathbf{m}_1$ developable surfaces of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. Finally, characterize one parameter family of the $\mathfrak{b}-\mathbf{m}_1$ developable surfaces in terms of their Bishop curvatures.

2. Riemannian Structure of Sol Space $\boldsymbol{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\boldsymbol{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ [11,12].

Note that the Sol metric can also be written as:

$$g_{\boldsymbol{Sol}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i,$$

where

$$\boldsymbol{\omega}^1 = e^z dx, \quad \boldsymbol{\omega}^2 = e^{-z} dy, \quad \boldsymbol{\omega}^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\boldsymbol{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\boldsymbol{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z)\,, \\ (x, y, z) &\to (x, y + c, z)\,, \\ (x, y, z) &\to \left(e^{-c}x, e^{c}y, z + c\right). \end{aligned}$$

3. Biharmonic New Type $\mathfrak{b}$–Slant Helices in Sol Space $\boldsymbol{Sol}^3$

Assume that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion [14,15] and

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{n}) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{b},\mathbf{b}) = 1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{n}) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{b}) = g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{b}) = 0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, [1]. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 + k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_1) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_2,\mathbf{m}_2) = 1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_1) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_2) = g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{t}, \mathbf{m}_1, \mathbf{m}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\delta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \delta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\delta(s), \\ k_2 &= \kappa(s)\sin\delta(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1e_1 + t^2e_2 + t^3e_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{3.5}$$

Theorem 3.1. *$\gamma : I \longrightarrow \boldsymbol{Sol}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \text{constant} \neq 0, \\ k_1'' - \left[k_1^2 + k_2^2\right]k_1 &= -k_1\left[2m_2^3 - 1\right] - 2k_2m_1^3m_2^3, \\ k_2'' - \left[k_1^2 + k_2^2\right]k_2 &= 2k_1m_1^3m_2^3 - k_2\left[2m_1^3 - 1\right]. \end{aligned} \tag{3.6}$$

Definition 3.2. *A regular curve $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ is called a new type slant helix provided the unit vector $\mathbf{m}_2$ of the curve γ has constant angle $\mathcal{M}$ with some fixed unit vector u,that is*

$$g_{\boldsymbol{Sol}^3}\left(\mathbf{m}_2\left(s\right), u\right) = \cos\mathcal{M} \text{ for all } s \in I. \tag{3.7}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that new type slant helices have unit speed. The new type slant helices can be identified by a simple condition on natural curvatures.

To separate a new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as new type $\mathfrak{b}-$slant helix.

We shall also use the following lemma.

Lemma 3.3. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed curve. Then γ is a new type $\mathfrak{b}-$slant helix if and only if*

$$k_1 = -k_2 \cot\mathcal{M}. \tag{3.8}$$

In the light of above theorem, we express the following result without proof:

Theorem 3.4. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the position vector of $\gamma$is*

$$\begin{aligned}\gamma\left(s\right) = & [\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\mathcal{S}_1\cos\left[\mathcal{S}_1 s+\mathcal{S}_2\right]+\sin\mathcal{M}\sin\left[\mathcal{S}_1 s+\mathcal{S}_2\right]]+\mathcal{S}_4 e^{-\sin\mathcal{M}s+\mathcal{S}_3}]\mathbf{e}_1 \\ & +[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\sin\mathcal{M}\cos\left[\mathcal{S}_1 s+\mathcal{S}_2\right]+\mathcal{S}_1\sin\left[\mathcal{S}_1 s+\mathcal{S}_2\right]]+\mathcal{S}_5 e^{\sin\mathcal{M}s-\mathcal{S}_3}]\mathbf{e}_2 \\ & +[-\sin\mathcal{M}s+\mathcal{S}_3]\mathbf{e}_3,\end{aligned} \tag{3.9}$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ are constants of integration, [7].

4. Inextensible Flows of $\mathfrak{b}-\mathbf{m}_1$ Developable Surfaces of Biharmonic New Type $\mathfrak{b}-$Slant Helices in Sol Space $\boldsymbol{Sol}^3$

To separate a $\mathbf{m}_1$ developable according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathfrak{b}-\mathbf{m}_1$ developable.

The purpose of this section is to study $\mathfrak{b}-\mathbf{m}_1$ developable of biharmonic new type $\mathfrak{b}-$slant helix in $\boldsymbol{Sol}^3$.

The $\mathfrak{b}-\mathbf{m}_1$ developable of γ is a ruled surface

$$\mathcal{D}_{new}\left(s, u\right) = \gamma\left(s\right) + u\mathbf{m}_1. \tag{4.1}$$

Definition 4.1. *A surface evolution $\mathcal{D}_{new}(s, u, t)$ and its flow $\frac{\partial\mathcal{D}_{new}}{\partial t}$ are said to be inextensible if its first fundamental form $\{\mathbf{E}, \mathbf{F}, \mathbf{G}\}$ satisfies*

$$\frac{\partial\mathbf{E}}{\partial t} = \frac{\partial\mathbf{F}}{\partial t} = \frac{\partial\mathbf{G}}{\partial t} = 0. \tag{4.2}$$

Definition 4.2. *We can define the following one-parameter family of developable ruled surface*

$$\mathcal{D}_{new}(s,u,t) = \gamma(s,t) + u\mathbf{m}_1(s,t). \tag{4.3}$$

Hence, we have the following theorem.

Theorem 4.3. *Let $\mathcal{D}_{new}$ be one-parameter family of the $\mathfrak{b}-\mathbf{m}_1$ developable of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then $\frac{\partial \mathcal{D}_{new}}{\partial t}$ is inextensible if and only if*

$$\begin{aligned} & \frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2 \\ & +\frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2 \\ = & -\frac{\partial}{\partial t}((1-uk_1(t))\sin\mathcal{M}(t))^2, \end{aligned} \tag{4.4}$$

Proof: Assume that $\mathcal{D}_{new}(s,u,t)$ be a one-parameter family of the $\mathfrak{b}-\mathbf{m}_1$ developable of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix.

From our assumption, we get the following equation

$$\begin{aligned} \mathbf{m}_2 = & \sin\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1 + \sin\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2 \\ & +\cos\mathcal{M}(t)\mathbf{e}_3, \end{aligned} \tag{4.5}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are smooth functions of time.

On the other hand, using Bishop formulas Eq.(3.3) and Eq.(2.1), we have

$$\mathbf{m}_1 = \cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1 - \sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2. \tag{4.6}$$

Using above equation and Eq.(4.5), we get

$$\mathbf{t} = \cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1 + \cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2 - \sin\mathcal{M}(t)\mathbf{e}_3. \tag{4.7}$$

Furthermore, we have the natural frame $\{(\mathcal{D}_{new})_s, (\mathcal{D}_{new})_u\}$ given by

$$\begin{aligned} (\mathcal{D}_{new})_s = & (1-uk_1(t))\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1 \\ & +(1-uk_1(t))\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2 - (1-uk_1(t))\sin\mathcal{M}(t) \end{aligned}$$

and

$$(\mathcal{D}_{new})_u = \cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1 - \sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2 \tag{4.8}$$

The components of the first fundamental form are

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} = & \frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2 \\ & +\frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2 \\ & +\frac{\partial}{\partial t}((1-uk_1(t))\sin\mathcal{M}(t))^2, \end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial t} &= 0, \\ \frac{\partial \mathbf{G}}{\partial t} &= 0.\end{aligned} \tag{4.9}$$

Hence, $\dfrac{\partial \mathcal{D}_{new}}{\partial t}$ is inextensible if and only if Eq.(4.4) is satisfied. This concludes the proof of theorem. □

Theorem 4.4. *Let $\mathcal{D}_{new}$ be one-parameter family of the $\mathfrak{b}-\mathbf{m}_1$ developable surface of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the parametric equations of this family are given by*

$$\begin{aligned}\boldsymbol{x}_{\mathcal{D}_{new}}(s,u,t) &= -\frac{e^{\sin \mathcal{M}(t)s-\mathcal{S}_3(t)}\cos \mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2 \mathcal{M}(t)}\mathcal{S}_1(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)] \\ &\quad +\frac{e^{\sin \mathcal{M}(t)s-\mathcal{S}_3(t)}\cos \mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2 \mathcal{M}(t)}\sin \mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]] \\ &\quad +\mathcal{S}_4(t)e^{-\sin \mathcal{M}(t)s+\mathcal{S}_3(t)}+u\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)], \qquad (4.10)\end{aligned}$$

$$\begin{aligned}\boldsymbol{y}_{\mathcal{D}_{new}}(s,u,t) &= -\sin \mathcal{M}(t)\frac{e^{-\sin \mathcal{M}(t)s+\mathcal{S}_3(t)}\cos \mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2 \mathcal{M}(t)}\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)] \\ &\quad +\mathcal{S}_1(t)\frac{e^{-\sin \mathcal{M}(t)s+\mathcal{S}_3(t)}\cos \mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2 \mathcal{M}(t)}\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]] \\ &\quad +\mathcal{S}_5(t)e^{\sin \mathcal{M}(t)s-\mathcal{S}_3(t)}-u\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)],\end{aligned}$$

$$\boldsymbol{z}_{\mathcal{D}_{new}}(s,u,t) = -\sin \mathcal{M}(t)s+\mathcal{S}_3(t),$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ are smooth functions of time.

Proof: The parametric equations of $\mathcal{D}_{new}(s,u,t)$ can be found from (3.9), (4.3). This concludes the proof of Theorem. □

We can use Mathematica in above theorem, yields

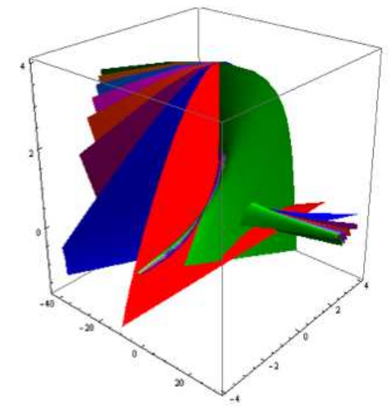

Fig. 1

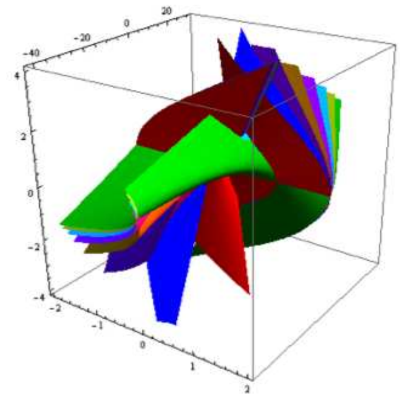

Fig. 2

Fig. 1,2: *The equation (4.10) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time* $t = 1$, $t = 1.2$, $t = 1.4$, $t = 1.6$, $t = 1.8$, $t = 2$, $t = 2.2$, $t = 2.4$, *respectively.*

Acknowledgments

The authors thank to the referee for useful suggestions and remarks for the revised version.

References

1. L. R. Bishop: *There is More Than One Way to Frame a Curve,* Amer. Math. Monthly 82 (3) (1975) 246-251.
2. R.M.C. Bodduluri, B. Ravani: *Geometric design and fabrication of developable surfaces,* ASME Adv. Design Autom. 2 (1992) , 243-250.

3. F. Dillen, W. Kuhnel: *Ruled Weingarten surfaces in Minkowski 3-space,* Manuscripta Math. 98 (1999), 307–320.

4. I. Dimitric: *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.

5. J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1–68.

6. J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

7. T. Körpınar and E. Turhan: *Biharmonic Slant Helices According to Bishop Frame in the Sol Space* $\mathfrak{sol}^3$, (in press)

8. T. Körpınar and E. Turhan: *On Spacelike Biharmonic New Type b-Slant Helices According to Bishop Frame in the Lorentzian Group of Rigid Motions* $\mathbb{E}(1,1)$, Bol. Soc. Paran. Mat. 30 (2) (2012), 91–100.

9. T. Körpınar and E. Turhan: *Biharmonic S-Curves According to Sabban Frame in Heisenberg Group Heis*^{3}, Bol. Soc. Paran. Mat. 31 (1) (2013), 205–211.

10. DY. Kwon , FC. Park, DP Chi: *Inextensible flows of curves and developable surfaces,* Appl. Math. Lett. 18 (2005), 1156-1162.

11. Y. Ou and Z. Wang: *Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces,* Mediterr. j. math. 5 (2008), 379–394

12. S. Rahmani: *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.

13. D. J. Struik: *Lectures on Classical Differential Geometry,* Dover, New-York, 1988.

14. E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis*^{3}, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

15. E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space* $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 99–104.

Mahmut Ergüt
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: mergut@firat.edu.tr

and

Talat Körpınar
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: talatkorpinar@gmail.com

and

Essin Turhan
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: essin.turhan@gmail.com