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σ-Ideals and Generalized Derivations in σ-Prime Rings

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abstract: Let R be a σ-prime ring and F and G be generalized derivations of R with associated derivations d and g respectively. In the present paper, we shall investigate the commutativity of R admitting generalized derivations F and G satisfying any one of the properties: (i) $F(x)y+F(y)x = xG(y)+yG(x)$, (ii) $F(x^2) = x^2$, (iii) $[F(x), y] = [x, G(y)]$, (iv) $d(x)F(y) = xy$, (v)$F([x, y]) = [F(x), y] + [d(y), x]$ and (vi) $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all x, y in some appropriate subset of R.

Key Words: Generalized derivations, σ-ideals, rings with involution, σ-prime rings

Contents

1. Introduction

Throughout, R will represent an associative ring with center $Z(R)$. Recall that a ring R is prime if a $Rb = 0$ implies $a = 0$ or $b = 0$. R is σ-prime if a $Rb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$ and R admits an involution σ. Every prime ring equipped with an involution is σ-prime but the converse need not be true in general. As an example, taking $S = R \times R^0$ where R^0 is an opposite ring of a prime ring R with $(x, y) = (y, x)$. Then S is not prime if $(0, a)S(a, 0) = 0$. But, R is σ-prime if we take $(a, b)S(x, y) = 0$ and $(a, b)S\sigma((x, y)) = 0$, then $aRx \times yRb = 0$ and $aRy \times xRb = 0$, and thus $aRx = yRb = aRy = xRb = 0$ (see for reference [9]). An ideal I of R is a σ-ideal if I is invariant under σ (viz:$\sigma(I) = I$). Oukhtite et al. [9] defined a set of symmetric and skew symmetric elements of R as $Sa_\sigma(R) = \{x \in R|\sigma(x) = \pm x\}$. For any $x, y \in R$ the symbol $[x, y]$ stands for commutator $xy - yx$ and $x \circ y$ denotes the anti-commutator $xy+yx$. We shall make extensive use of the basic commutator identities as follows:

$[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = y[x, z] + [x, y]z$, $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$ and $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$. As defined by Bresar [6], an additive map $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ (an additive map $d : R \to R$ is a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$) such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. One can easily check that the notion of generalized derivation

2000 *Mathematics Subject Classification:* 16W10, 16W25, 16U80 covers the notions of a derivation and a left multiplier (i.e. $F(xy) = F(x)y$ for all $x, y \in R$). Particularly, one can observe that, for a fixed $a \in R$, the map $d_a : R \to R$ defined by $d_a(x) = [a, x]$ for all $x \in R$ is a derivation which is said to be an inner derivation. An additive map $g_{a,b} : R \to R$ is called a generalized inner derivation if $g_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$.

It is easy to see that if $g_{a,b}(x)$ is a generalized inner derivation, then $g_{a,b}(xy) = g_{a,b}(x)y + xd_{-b}(y)$ for all $x, y \in R$, where d_{-b} is an inner derivation.

Several authors [1,2,3,17,18,19,20] have established numerous results concerning derivations and generalized derivations of prime rings. In 2005, Oukhtite et al. conferred an extension of prime rings in the form of σ-prime rings and proved a number of results which hold true for prime rings (see for references [9,10,11,12, 13,14,15,16]). In [7] and [8] author et al. extended results concerning derivations and generalized derivations of σ-prime rings to some more general settings. Ashraf et al. too contributed to this newly emerged theory in [5], apart from great deal of work in the field of prime rings.

Recently, Ashraf et al. [4] extended some known theorems for derivations to generalized derivations in the setting of semiprime rings. In this context, a natural question arises: Under what additional conditions the above results can be extended to σ-prime (σ-semiprime) rings. However, in this perspective, we prove the results for σ-prime rings exhibiting generalized derivations F and G associated with derivations d and g respectively and hope for similar conversion to σ-semiprime rings in near future. Now, let I be σ-ideal of σ-prime ring R. For every $x, y \in I$, we define the following properties.

(P_1) $(F(x)y + F(y)x) \pm (xG(y) + yG(x)) = 0$.

(P_2) $F(x^2) \pm x^2 = 0$.

(P_3) $[F(x), y] \pm [x, G(y)] = 0$.

(P_4) $d(x)F(y) \pm xy = 0$.

(P_5) $F([x, y]) = [F(x), y] + [d(y), x]$.

(P_6) $F(x) \circ y - d(y) \circ x = 0$.

2. Main Results

In order to prove our results, we need the following known lemmas:

Lemma 2.1 ([10, Lemma 3.1]). *Let R be a σ-prime ring and let I be a nonzero σ-ideal of R. If a, b in R satisfy $aIb = aI\sigma(b) = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([11, Lemma 2.2]). *Let I be a nonzero σ-ideal of R and $0 \neq d$ be a derivation on R which commutes with σ. If $[x, R]Id(x) = 0$ for all $x \in I$, then R is commutative.*

We begin with

Theorem 2.3. *Let R be a 2-torsion free σ-prime ring and I a nonzero σ-ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g which commutes with σ. If R satisfies one of the properties (P_1) and (P_3), then R is commutative.*

Proof: (i) By the hypothesis (P_1), we have

$$F(x)y + F(y)x = xG(y) + yG(x) \quad \text{for all } x, y \in I. \tag{2.1}$$

Combining the expressions obtained after replacing x by xy in (2.1) and multiplying (2.1) with y from the right, we get

$$xd(y)y = yxg(y) + x[y, G(y)] \quad \text{for all } x, y \in I. \tag{2.2}$$

For any $r \in R$, replacing x by rx in (2.2) and combining with the expression obtained by multiplying (2.2) with r from the left, we get

$$[y, r]xg(y) = 0.$$

Therefore,

$$[y, r]Ig(y) = 0 \quad \text{for all } x \in I. \tag{2.3}$$

Since I is a σ-ideal and $g\sigma = \sigma g$, for all $y \in I \cap Sa_\sigma(R)$, so in view of Lemma 2.1, we have $[y, r] = 0$ or $g(y) = 0$. Using the fact that $y + \sigma(y) \in Sa_\sigma(R) \cap I$ for all $y \in I$, then $[y + \sigma(y), r] = 0$ or $g(y + \sigma(y)) = 0$ for all $y \in I$ and $r \in R$. Now, two cases arise.

Case 1: If $[y + \sigma(y), r] = 0$ and $y - \sigma(y) \in Sa_\sigma(R) \cap I$,yields $[y - \sigma(y), r] = 0$ or $g(y - \sigma(y)) = 0$ $r \in R$.

If $[y - \sigma(y), r] = 0$ then $0 = [y - \sigma(y), r] + [y + \sigma(y), r] = 2[y, r] = 0$ implies $[y, r] = 0$, since char $R \neq 2$. If $g(y - \sigma(y)) = 0$ $r \in R$, then $g(y) = g(\sigma(y)) = \sigma(g(y))$.

An application of Lemma 2.1 equation (2.3) implies $[y, r] = 0$ or $g(y) = 0$.

Case 2: If $g(y + \sigma(y)) = 0$, then $g(y) = -g(\sigma(y)) = -\sigma(g(y))$, and in view of (2.3)

$$[y, r]Ig(y) = 0 = [y, r]I\sigma(g(y)).$$

By Lemma 2.1, we arrive at $[y, r] = 0$ or $g(y) = 0$.

If $g(y) = 0$, then for any r in R, we find that $yd(r) = 0$ for all $y \in I$. Hence,

$$Id(r) = IRd(r) = \sigma(I)Rd(r) = 0.$$

Since $I \neq 0$ and R is a σ-prime, we obtain $d(R) = 0$, (i.e. $d = 0$) yields a contradiction.

Next, suppose that $[y, r] = 0$. Then for any s in R, we have

$$0 = [sy, r] = [s, r]y = [s, r]I = [s, r]RI = [s, r]R\sigma(I) = 0.$$

Since $I \neq 0$ and R is σ-prime, we obtain $[s, r] = 0$ for all $r, s \in R$. Hence R is commutative.

(ii) Similarly we can prove that R is commutative, if R satisfies (P_3). □

Remark 2.4. *Taking $G = F$ or $G = -F$ in the hypothesis of Theorem 2.4, we get the following.*

Corollary 2.5. *Let R be a 2-torsion free σ-prime ring and I a nonzero σ-ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d which commutes with σ, such that $[F(x), y] + [F(y), x] = 0$ for all $x, y \in I$ or if $F(x) \circ y + F(y) \circ x = 0$ for all $x, y \in I$, then R is commutative.*

Theorem 2.6. *Let R be a 2-torsion free σ-prime ring and I a nonzero σ-ideal of R. Suppose that R admits generalized derivations F with associated nonzero derivation d which commutes with σ such that the property (P_2) or (P_4) is satisfied. Then R is commutative.*

Proof: From the hypothesis of (P_2), we write

(i) $F(x^2) = x^2$ for all $x \in I$. Replacing x by $x + y$ in the above relation and using (P_2), we obtain

$$F(x \circ y) = x \circ y \quad \text{for all} \quad x, y \in I.$$

Using Theorem 2.2 of [14], we get the required result.

(ii) $F(x^2) + x^2 = 0$ for all $x \in I$, then as (i) we get $F(x \circ y) + (x \circ y) = 0 \forall x, y \in I$. Following the same technique as used in the proof of [14, Theorem 2.2], we get the required result. □

Corollary 2.7. *Let R be a 2-torsion free σ-prime ring and I be a nonzero σ-ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g which commutes with σ. If $[F(x), y] = [x, F(y)]$ for all $x, y \in I$ (or $[F(x), y] + [x, F(y)] = 0$) for all $x, y \in I$, then R is commutative.*

Theorem 2.8. *Let R be a 2-prime ring and I be a nonzero σ-ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d commuting with σ such that property (P_5) or (P_6) is satisfied. Then R is commutative.*

Proof: By our hypothesis (P_5), we have

$$F([x, y]) = [F(x), y] + [d(y), x]. \tag{2.4}$$

Replacing y by yx in (2.4) and employing (2.4), we find that

$$2[x, y]d(x) = y[F(x), x] + y[d(x), x] \quad \text{for all } x, y \in I. \tag{2.5}$$

For any $r \in R$, putting y by ry in (2.5) and applying (2.5), we get

$$2[x, r]yd(x) = 0 \quad \text{for all } x, y \in I.$$

Since R is 2-torsion free, we get $[x, r]yd(x) = 0$ for all $x, y \in I$ and $r \in R$.

Therefore, $[x, R]Id(x) = 0$ for all $x \in I$ and $r \in R$.

By application of Lemma 2.2, we conclude that R is commutative. □

3. Counter-examples

Remark 3.1. *The following example shows that R to be prime is essential in the hypothesis of our theorems.*

Example 3.2. *Take any arbitrary ring M and $R = \left\{ f \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in M \right\}$ a non commutative prime ring and $I = \left\{ f \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in M \right\}$ be a nonzero ideal of R.*

Define a map $F : R \to R$ by $F(x) = 2 \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Then it is obvious to see that F is a generalized derivation associated with a nonzero derivation $d(x) = [e_{11}, x]$. Clearly, F satisfies the properties (P_1-P_6), for example $F([x, y]) = [x, y]$ for all $x, y \in I$. However, R is not commutative.

Example 3.3. *Take $M = Z[X] \times Z[X]$; if we define an addition on M by component wise and multiplication by $(p_1, p_2)(q_1, q_2) = (p_1q_2 - p_2q_1, 0)$, then M is a ring such that $m = 0$ for all $m \in M$. Moreover, M is non commutative and $mn = -nm$ for all $m, n \in M$. Let F be the additive mapping defined on the ring $R = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \setminus a, b \in M \right\}$ by $F \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ b - a & a \end{pmatrix}$. Clearly, F is a nontrivial left multiplier of R (i.e. derivation $d = 0$). Since $mn = -nm$ for all $m, n \in M$, it is easy to check that the map $\sigma : R \to R$ defined by $\sigma \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ -b & -a \end{pmatrix}$ is an involution.*

On the other hand, if we set $a = \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \in R$, where $m = 0$, then $aRa = 0$.

And $aR\sigma(a) = 0$; proving that R is a non σ-prime ring.

Let $U = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \middle/ b \in M \right\}$.

It is clear that U is a σ-Lie ideal of R such that $F([u, v]) = [u, v]$ for all $u, v \in U$. Moreover, if $m, n \in M$ are such that $mn = 0$, then $u = \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \in U$ and $r = \begin{pmatrix} 0 & 0 \\ m & n \end{pmatrix} \in R$ and $[u, r] = 0$, proving that $U \subseteq Z(R)$. Accordingly, in Theorem 2.8 the hypothesis of σ- primeness is crucial.

Remark 3.4. *The following examples show that the property of primeness in the stated results cannot be omitted. (i) Let R be a prime ring and $d_{1,}d_2$ be derivations of R such that at least one is non-zero. If $d_1(x)x + xd_2(x) = 0$ for all $x \in R$, then R is commutative; (ii) If a prime ring R has a non-zero commuting derivation on itself, then R is commutative.*

Example 3.5. *Let S be a ring in which $a^2 = 0$, $a \in S$ and $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle/ a, b \in S \right\}$.*

Define $d_1 : R \to R$ *by* $d_1 \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ *and* $d_2 : R \to R$ *by* $d_2 \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a-b & 0 \end{pmatrix}$.

Then R *is a ring under the usual operations. Clearly,* d_1 *and* d_2 *are derivations of* R *such that* $d_1(x)x + xd_2(x) = 0$*. This indicates that the hypothesis of primness in not superfluous.*

Remark 3.6. *Example 3.3 demonstrates that if we replace the prime ring by a semi prime ring in Remark 3.4 (ii), then R may not be commutative, even for an ordinary derivation.*

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