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A Bertrand Postulate for a Subclass of Primes

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Abstract: Let d be a squarefree integer and consider the subclass of primes with Legendre symbol $(\frac{d}{p}) = +1$. It is shown that for x large enough $(x, 2x]$ contain a prime of this type.

Key Words: Primes ; Legendre Symbol ; Bertrand's Postulate.

Bertrand's Postulate states that"for every $n > 1$ there is at least one prime p such that $n < p < 2n$".

Let d be a squarefree integer. It is known ([2],p75-76) that the set of primes p with Legendre symbol $(\frac{d}{p}) = +1$ has (analytic/natural) density $\frac{1}{2}$. We state this as

Lemma 1. *Let* $\pi_1(x) = |\{p|p \le x, p \quad prime, (\frac{d}{p}) = +1\}|$. *Then*

$$\lim_{x\to\infty} \frac{\pi_1(x)}{\pi(x)} = \frac{1}{2}$$

Here $\pi(x) = \sum_{p\le x} 1$ *is the usual counting function. We prove the following using certain standard results via Lemmas 1,2,3.*

Proposition 1. *For all* x *large enough, the interval* $(x, 2x]$ *contains a prime* p *with* $(\frac{d}{p}) = +1$.

Remark 1. *Unlike Bertrand's postulate, such a statement can fail for small* x*, even if the interval is "doubled" to* $(x, 4x)$*. For example if* $d = 5$ *and* $x = 2$*, then* $(2, 8)$ *contains three primes; 3,5 and 7. But* $(\frac{5}{3}) = -1$, $(\frac{5}{5}) = 0$ *and* $(\frac{5}{7}) = -1$.
Recall Chebyshev's function

$$\theta(\mathrm{x}) = \sum_{p\le x} logp = log(\prod_{p\le x} p)$$

We introduce correspondingly $\theta_1(\mathrm{x}) = \sum_{p\le x,(\frac{d}{p})=+1} logp$. *Note that* $\pi_1(x) \le \pi(x)$ *and* $\theta_1(\mathrm{x}) \le \theta(\mathrm{x})$

Lemma 2. $\lim_{x\to\infty} \frac{\theta_1(\mathrm{x})}{x} = \frac{1}{2}$

Proof: $\lim_{x\to\infty} \frac{\theta_1(\mathrm{x})}{x} = \lim_{x\to\infty} [\frac{\pi_1(x) logx}{x} - \frac{1}{x}\int_2^x \frac{\pi_1(t)}{t} dt]$ by adapting directly the proof of the corresponding result for θ and π ([1], Th 4.3).

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Again $\lim_{x\to\infty} \frac{1}{x}\int_2^x \frac{\pi_1(t)}{t}dt = 0$ ([1],p79).This forces the second term above to tend to 0. Hence

$$\lim_{x\to\infty} \frac{\theta_1(\mathrm{x})}{x} = \lim_{x\to\infty} \frac{\pi_1(x) log x}{x} = \lim_{x\to\infty} \frac{\pi_1(x)}{\pi(x)}$$
$$= \frac{1}{2}$$

by the Prime Number Theorem ($\pi(x) \sim \frac{log x}{x}$) and Lemma1 □

Lemma 3. $\lim_{x\to\infty}(\frac{\theta_1(2\mathrm{x})}{x} - \frac{\theta_1(\mathrm{x})}{x}) = 2(\frac{1}{2}) - \frac{1}{2} = \frac{1}{2}$

Proof: Apply Lemma2 to each of the limits. □

Proof of Proposition1:

$$\theta_1(2\mathrm{x}) - \theta_1(\mathrm{x}) = log(\prod_{p\leq 2x, (\frac{d}{p}=+1)} p) - (log \prod_{p\leq x, (\frac{d}{p})=+1} p)$$

$$\therefore \theta_1(2\mathrm{x}) - \theta_1(\mathrm{x}) = log(\prod_{x<p\leq 2x, (\frac{d}{p}=+1)} p)$$

This is zero precisely when $(x, 2x]$ does not contain any prime p with the symbol +1. But if it is zero for infinitely many x, with $x \to \infty$, we have a contradiction to Lemma3 as there would be a subsequence with limit $0 \neq \frac{1}{2}$. Hence there is x_0 such that for all $x > x_0$, $(x, 2x]$ contains a prime p with symbol $(\frac{d}{p}) = +1$.

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