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Inextensible Flows of Spacelike Curves on Spacelike Surfaces according to Darboux Frame in $\mathbb{M}_1^3$

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ABSTRACT: In this paper, we study inextensible flows of spacelike curves on oriented spacelike surfaces in $\mathbb{M}_1^3$. We give necessary and sufficient conditions for inextensible flows of spacelike curves on oriented spacelike surfaces in $\mathbb{M}_1^3$.

Key Words: Inextensible flows, Darboux Frame, Curvatures.

Contents

1. Introduction

Construction of fluid flows constitutes an active research field with a high industrial impact. Corresponding real-world measurements in concrete scenarios complement numerical results from direct simulations of the Navier-Stokes equation, particularly in the case of turbulent flows, and for the understanding of the complex spatio-temporal evolution of instationary flow phenomena. More and more advanced imaging devices (lasers, highspeed cameras, control logic, etc.) are currently developed that allow to record fully timeresolved image sequences of fluid flows at high resolutions. As a consequence, there is a need for advanced algorithms for the analysis of such data, to provide the basis for a subsequent pattern analysis, and with abundant applications across various areas.

This study is organised as follows: Firstly, we study inextensible flows of spacelike curves on oriented spacelike surfaces in $\mathbb{M}_1^3$. Finally, we give necessary and sufficient conditions for inextensible flows of spacelike curves on oriented spacelike surfaces in $\mathbb{M}_1^3$.

2. Preliminaries

The Minkowski 3-space $\mathbb{M}_1^3$ provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{M}_1^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{M}_1^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the spacelike curve γ in the space $\mathbb{M}_1^3$. For an arbitrary spacelike curve γ with first and second curvature, κ and τ in the space $\mathbb{M}_1^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= \kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' &= \tau\mathbf{N}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \langle\mathbf{T}, \mathbf{T}\rangle &= 1, \langle\mathbf{N}, \mathbf{N}\rangle = -1, \langle\mathbf{B}, \mathbf{B}\rangle = 1, \\ \langle\mathbf{T}, \mathbf{N}\rangle &= \langle\mathbf{T}, \mathbf{B}\rangle = \langle\mathbf{N}, \mathbf{B}\rangle = 0. \end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s)$ and $\tau = \tau(s)$.

Torsion of the spacelike curve γ is given by the aid of the mixed product

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

A surface M in the Minkowski 3-space $\mathbb{M}_1^3$ is said to be space-like, time-like surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [9].

If the surface $\mathcal{M}$ is an oriented spacelike surface, then the curve $\alpha(s)$ lying on $\mathcal{M}$ is a spacelike curve. Thus, the equations which describe the Darboux frame of $\alpha(s)$ is given by :

$$\begin{aligned} \mathbf{T}' &= \kappa_g\mathbf{P} + \kappa_n\mathbf{n}, \\ \mathbf{P}' &= -\kappa_g\mathbf{T} + \tau_g\mathbf{n}, \\ \mathbf{n}' &= \kappa_n\mathbf{T} + \tau_g\mathbf{P}, \end{aligned} \tag{2.2}$$

where $\mathbf{T}, \mathbf{P}$,$\mathbf{n}$ satisfy the following properties:

$$\begin{aligned} &< \mathbf{T}, \mathbf{T} >= 1, \ < \mathbf{n}, \mathbf{n} > = -1, \ < \mathbf{P}, \mathbf{P} > = 1, \\ &< \mathbf{T}, \mathbf{n} > =< \mathbf{T}, \mathbf{P} > =< \mathbf{n}, \mathbf{P} > = 0. \end{aligned}$$

In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface $\mathcal{M}$ and $\mathbf{P}$ is a unit vector given by $\mathbf{P} = \mathbf{n} \times \mathbf{T}$.

3. Inextensible Flows of Spacelike Curves on Spacelike Surface according to Darboux Frame in $\mathbb{M}_1^3$

Let $\alpha(u,t)$ is a one parameter family of smooth spacelike curves in $\mathbb{M}_1^3$.
The arclength of α is given by

$$s(u) = \int_0^u \left|\frac{\partial\alpha}{\partial u}\right| du, \tag{3.1}$$

where

$$\left|\frac{\partial\alpha}{\partial u}\right| = \left|\left\langle\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial u}\right\rangle\right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{\nu}\frac{\partial}{\partial u},$$

where $v = \left|\frac{\partial\alpha}{\partial u}\right|$ and the arclength parameter is $ds = vdu$.

Any flow of α can be represented as

$$\frac{\partial\alpha}{\partial t} = \mathcal{A}_1^{\mathcal{D}}\mathbf{T} + \mathcal{A}_2^{\mathcal{D}}\mathbf{P} + \mathcal{A}_3^{\mathcal{D}}\mathbf{n}, \tag{3.3}$$

where $\mathcal{A}_1^{\mathcal{D}}, \mathcal{A}_2^{\mathcal{D}}, \mathcal{A}_3^{\mathcal{D}}$ are smooth functions.

Letting the arclength variation be

$$s(u,t) = \int_0^u vdu.$$

In the $\mathbb{M}_1^3$ the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial v}{\partial t}du = 0, \tag{3.4}$$

for all $u \in [0,l]$.

Definition 3.1. *Let $\mathcal{M}$ be an oriented spacelike surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. The flow $\frac{\partial\alpha}{\partial t}$ on $\mathcal{M}$ are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial\alpha}{\partial u}\right| = 0.$$

Lemma 3.2. *Let $\mathcal{M}$ be an oriented spacelike surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. The flow* $\frac{\partial \alpha}{\partial t} = \mathcal{A}_1^{\mathcal{D}}\mathbf{T} + \mathcal{A}_2^{\mathcal{D}}\mathbf{P} + \mathcal{A}_3^{\mathcal{D}}\mathbf{n}$ *is inextensible if and only if*

$$\frac{\partial v}{\partial t} - \frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial u} = -\mathcal{A}_2^{\mathcal{D}} v\kappa_g + \mathcal{A}_3^{\mathcal{D}} v\kappa_n. \tag{3.5}$$

Proof: Suppose that $\frac{\partial \alpha}{\partial t}$ be a smooth flow of the spacelike curve α. Using definition of α, we have

$$v^2 = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle. \tag{3.6}$$

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since and are independent coordinates.
Further differentiation of (3.6) gives

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, we have

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial \alpha}{\partial t}) \right\rangle.$$

From (3.3), we obtain

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u}\left(\mathcal{A}_1^{\mathcal{D}}\mathbf{T} + \mathcal{A}_2^{\mathcal{D}}\mathbf{P} + \mathcal{A}_3^{\mathcal{D}}\mathbf{n}\right) \right\rangle.$$

By the formula of the Darboux, we have

$$\frac{\partial v}{\partial t} =< \mathbf{T}, \left(\frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial u} - \mathcal{A}_2^{\mathcal{D}} v\kappa_g + \mathcal{A}_3^{\mathcal{D}} v\kappa_n\right)\mathbf{T} + \left(\mathcal{A}_1^{\mathcal{D}} v\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial u} + \mathcal{A}_3^{\mathcal{D}} v\tau_g\right)\mathbf{P}$$
$$+ \left(\mathcal{A}_1^{\mathcal{D}} v\kappa_n + \mathcal{A}_2^{\mathcal{D}} v\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial u}\right)\mathbf{n} >.$$

Making necessary calculations from above equation, we have (3.5), which proves the lemma. □

Theorem 3.3. *Let $\mathcal{M}$ be an oriented spacelike surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. The flow* $\frac{\partial \alpha}{\partial t}$ *is inextensible if and only if*

$$\frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial u} = \mathcal{A}_2^{\mathcal{D}} v\kappa_g - \mathcal{A}_3^{\mathcal{D}} v\kappa_n. \tag{3.7}$$

Proof: Assume that $\frac{\partial \alpha}{\partial t}$ be inextensible.

By Definition 3.1, the flow $\frac{\partial \alpha}{\partial t}$ is inextensible if and only if

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}du=\int_0^u\left(\frac{\partial \mathcal{A}_1^{\mathfrak{D}}}{\partial u}-\mathcal{A}_2^{\mathfrak{D}}v\kappa_g+\mathcal{A}_3^{\mathfrak{D}}v\kappa_n\right)du=0. \tag{3.8}$$

Substituting (3.5) in (3.8) complete the proof of the theorem. □

We now restrict ourselves to arc length parametrized curves. That is, $v=1$ and the local coordinate u corresponds to the curve arc length s. We require the following lemma.

Lemma 3.4.

$$\frac{\partial \mathbf{T}}{\partial t} = \left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}+\mathcal{A}_3^{\mathfrak{D}}\tau_g\right)\mathbf{P}+\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)\mathbf{n}, \tag{3.9}$$

$$\frac{\partial \mathbf{P}}{\partial t} = -\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}+\mathcal{A}_3^{\mathfrak{D}}\tau_g\right)\mathbf{T}+\psi\mathbf{n}, \tag{3.10}$$

$$\frac{\partial \mathbf{n}}{\partial t} = \left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)\mathbf{T}-\psi\mathbf{P}, \tag{3.11}$$

where $\psi=\left\langle \frac{\partial \mathbf{P}}{\partial t},\mathbf{n}\right\rangle$.

Proof: Using definition of α, we have

$$\frac{\partial \mathbf{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \alpha}{\partial s}=\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathfrak{D}}\mathbf{T}+\mathcal{A}_2^{\mathfrak{D}}\mathbf{P}+\mathcal{A}_3^{\mathfrak{D}}\mathbf{n}).$$

Using the Darboux equations, we have

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial t} &= \left(\frac{\partial \mathcal{A}_1^{\mathfrak{D}}}{\partial s}-\mathcal{A}_2^{\mathfrak{D}}\kappa_g+\mathcal{A}_3^{\mathfrak{D}}\kappa_n\right)\mathbf{T}+\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}+\mathcal{A}_3^{\mathfrak{D}}\tau_g\right)\mathbf{P}\\ &\quad+\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)\mathbf{n}.\end{aligned} \tag{3.12}$$

Thus, we rewrite (3.12) as follows:

$$\frac{\partial \mathbf{T}}{\partial t}=\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}+\mathcal{A}_3^{\mathfrak{D}}\tau_g\right)\mathbf{P}+\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)\mathbf{n}.$$

Considering (3.12), we have

$$\begin{aligned}
\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g + \left\langle \mathbf{T}, \frac{\partial \mathbf{P}}{\partial t} \right\rangle &= 0, \\
-\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g - \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s} + \left\langle \mathbf{T}, \frac{\partial \mathbf{n}}{\partial t} \right\rangle &= 0, \\
\psi + \left\langle \mathbf{P}, \frac{\partial \mathbf{n}}{\partial t} \right\rangle &= 0.
\end{aligned}$$

Then, a straightforward computation using above system gives

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial t} &= -\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\mathbf{T} + \psi\mathbf{n}, \\
\frac{\partial \mathbf{n}}{\partial t} &= \left(\mathcal{A}_1^{\mathcal{D}}\kappa_n + \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{T} - \psi\mathbf{P},
\end{aligned}$$

where $\psi = \left\langle \frac{\partial \mathbf{P}}{\partial t}, \mathbf{n} \right\rangle$.

Thus, we obtain the theorem. □

The following theorem states the conditions on the curvature and torsion for the flow to be inextensible.

Theorem 3.5. *Let $\mathcal{M}$ be an oriented spacelike surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. If $\frac{\partial \alpha}{\partial t}$ is inextensible, then the following system of partial differential equations holds:*

$$\begin{aligned}
\frac{\partial \kappa_g}{\partial t} - \kappa_n\psi &= \frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_g) + \frac{\partial^2 \mathcal{A}_2^{\mathcal{D}}}{\partial s^2} + \frac{\partial}{\partial s}(\mathcal{A}_3^{\mathcal{D}}\tau_g) + \mathcal{A}_1^{\mathcal{D}}\kappa_n\tau_g + \mathcal{A}_2^{\mathcal{D}}\tau_g^2 + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\tau_g, \\
\frac{\partial \kappa_n}{\partial t} + \kappa_g\psi &= \frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_n) + \frac{\partial}{\partial s}(\mathcal{A}_2^{\mathcal{D}}\tau_g) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2} + \mathcal{A}_1^{\mathcal{D}}\kappa_g\tau_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\tau_g + \mathcal{A}_3^{\mathcal{D}}\tau_g^2.
\end{aligned}$$

Proof: Using (3.9), we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{T}}{\partial t} &= \frac{\partial}{\partial s}\left[\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\mathbf{P} + \left(\mathcal{A}_1^{\mathcal{D}}\kappa_n + \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{n}\right] \\
&= [\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n + \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\kappa_n - \left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\kappa_g]\mathbf{T} \\
&\quad + \left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_g) + \frac{\partial^2 \mathcal{A}_2^{\mathcal{D}}}{\partial s^2} + \frac{\partial}{\partial s}(\mathcal{A}_3^{\mathcal{D}}\tau_g) + \mathcal{A}_1^{\mathcal{D}}\kappa_n\tau_g + \mathcal{A}_2^{\mathcal{D}}\tau_g^2 + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\tau_g\right)\mathbf{P} \\
&\quad + \left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_n) + \frac{\partial}{\partial s}(\mathcal{A}_2^{\mathcal{D}}\tau_g) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2} + \mathcal{A}_1^{\mathcal{D}}\kappa_g\tau_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\tau_g + \mathcal{A}_3^{\mathcal{D}}\tau_g^2\right)\mathbf{n}.
\end{aligned}$$

On the other hand, from Darboux frame we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{T}}{\partial s} &= \frac{\partial}{\partial t}(\kappa_g \mathbf{P}+\kappa_n \mathbf{n}) \\ &= -[\kappa_g\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}+\mathcal{A}_3^{\mathfrak{D}}\tau_g\right)-\kappa_n\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)]\mathbf{T} \\ &\quad +\left[\kappa_g\psi+\frac{\partial \kappa_n}{\partial t}\right]\mathbf{n}+\left[\frac{\partial \kappa_g}{\partial t}-\kappa_n\psi\right]\mathbf{P}.\end{aligned}$$

Thus,

$$\frac{\partial \kappa_n}{\partial t}+\kappa_g\psi=\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathfrak{D}}\kappa_n)+\frac{\partial}{\partial s}(\mathcal{A}_2^{\mathfrak{D}}\tau_g)+\frac{\partial^2 \mathcal{A}_3^{\mathfrak{D}}}{\partial s^2}+\mathcal{A}_1^{\mathfrak{D}}\kappa_g\tau_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}\tau_g+\mathcal{A}_3^{\mathfrak{D}}\tau_g^2$$

and

$$\frac{\partial \kappa_g}{\partial t}-\kappa_n\psi=\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathfrak{D}}\kappa_g)+\frac{\partial^2 \mathcal{A}_2^{\mathfrak{D}}}{\partial s^2}+\frac{\partial}{\partial s}(\mathcal{A}_3^{\mathfrak{D}}\tau_g)+\mathcal{A}_1^{\mathfrak{D}}\kappa_n\tau_g+\mathcal{A}_2^{\mathfrak{D}}\tau_g^2+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\tau_g.$$

Thus, we obtain the theorem. □

Corollary 3.6.

$$-\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g\tau_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}\tau_g+\mathcal{A}_3^{\mathfrak{D}}\tau_g^2\right)+\frac{\partial \kappa_n}{\partial t}=-\left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathfrak{D}}\kappa_n)+\frac{\partial}{\partial s}\left(\mathcal{A}_2^{\mathfrak{D}}\tau_g\right)+\frac{\partial^2 \mathcal{A}_3^{\mathfrak{D}}}{\partial s^2}\right)+\kappa_g\psi.$$

Proof: Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{n}}{\partial t} &= \frac{\partial}{\partial s}\left[-\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)\mathbf{T}-\psi\mathbf{P}\right] \\ &= [-\left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathfrak{D}}\kappa_n)+\frac{\partial}{\partial s}\left(\mathcal{A}_2^{\mathfrak{D}}\tau_g\right)+\frac{\partial^2 \mathcal{A}_3^{\mathfrak{D}}}{\partial s^2}\right)+\kappa_g\psi]\mathbf{T} \\ &\quad +[-\kappa_g\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)-\frac{\partial \psi}{\partial s}]\mathbf{P} \\ &\quad -[\kappa_n\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}+\tau_g\psi\right)]\mathbf{n}.\end{aligned}$$

On the other hand, a straight forward computation gives

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{n}}{\partial s} &= [-\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g\tau_g+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}\tau_g+\mathcal{A}_3^{\mathfrak{D}}\tau_g^2\right)+\frac{\partial \kappa_n}{\partial t}]\mathbf{T} \\ &\quad +[\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_g\kappa_n+\frac{\partial \mathcal{A}_2^{\mathfrak{D}}}{\partial s}\kappa_n+\mathcal{A}_3^{\mathfrak{D}}\kappa_n\tau_g\right)+\frac{\partial \tau_g}{\partial t}]\mathbf{P} \\ &\quad +\kappa_n\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathfrak{D}}}{\partial s}+\psi\tau_g\right)\mathbf{n}.\end{aligned}$$

Combining these we obtain the corollary. □

In the light of Theorem 3.5, we express the following lemmas without proofs:

Lemma 3.7.

$$\mathcal{A}_1^{\mathfrak{D}}\kappa_g\kappa_n+\frac{\partial\mathcal{A}_2^{\mathfrak{D}}}{\partial s}\kappa_n+\mathcal{A}_3^{\mathfrak{D}}\kappa_n\tau_g+\frac{\partial\tau_g}{\partial t}=-\kappa_g\left(\mathcal{A}_1^{\mathfrak{D}}\kappa_n+\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial\mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)-\frac{\partial\psi}{\partial s}.$$

Corollary 3.8. *Let $\mathcal{M}$ be an oriented spacelike surface, α lying on $\mathcal{M}$ and the flow $\frac{\partial\alpha}{\partial t}$ is inextensible in Minkowski 3-space $\mathbb{M}_1^3$. If α is a geodesic curve, then*

$$\frac{\partial\psi}{\partial s}=-\frac{\partial\mathcal{A}_2^{\mathfrak{D}}}{\partial s}\kappa_n-\mathcal{A}_3^{\mathfrak{D}}\kappa_n\tau_g-\frac{\partial\tau_g}{\partial t}.$$

Proof: By using $\kappa_g=0$ in Lemma 3.7, we get above equation. This completes the proof. □

Corollary 3.9. *Let $\mathcal{M}$ be an oriented spacelike surface, α lying on $\mathcal{M}$ and the flow $\frac{\partial\alpha}{\partial t}$ is inextensible in Minkowski 3-space $\mathbb{M}_1^3$. If α is a principal line, then*

$$\frac{\partial\psi}{\partial s}\kappa_g+\frac{\partial\mathcal{A}_3^{\mathfrak{D}}}{\partial s}+\frac{\partial\mathcal{A}_2^{\mathfrak{D}}}{\partial s}\kappa_n+\frac{\partial\tau_g}{\partial t}=-\mathcal{A}_1^{\mathfrak{D}}\kappa_g\kappa_n-\kappa_g\mathcal{A}_1^{\mathfrak{D}}\kappa_n,$$

Proof: Substituting $\tau_g=0$ in Lemma 3.7, we get above equation. This completes the proof. □

Corollary 3.10. *Let $\mathcal{M}$ be an oriented spacelike surface, α lying on $\mathcal{M}$ and the flow $\frac{\partial\alpha}{\partial t}$ is inextensible in Minkowski 3-space $\mathbb{M}_1^3$. If α is a asymptotic line, then*

$$\kappa_g\left(\mathcal{A}_2^{\mathfrak{D}}\tau_g+\frac{\partial\mathcal{A}_3^{\mathfrak{D}}}{\partial s}\right)+\frac{\partial\psi}{\partial s}=0.$$

Proof: By using $\kappa_n=0$ in Lemma 3.7, we get above equation. This completes the proof. □

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