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Decay of Small Solutions for the Zakharov-Kuznetsov Equation posed on a half-strip

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ABSTRACT: We formulate in a half-strip an initial boundary value problem for the Zakharov-Kuznetsov equation. Assuming the existence of a regular global solution, we prove an exponential decay for small initial data.

Key Words: Zakharov-Kuznetsov equation , Exponential Decay

Contents

1. Introduction

Dispersive equations attract attention of many mathematicians. More popular are Korteweg-de Vries and Schrödinger equations. The theory of the Cauchy problem for them nowadays is well developed and presented in papers of Bona and his colleagues [1], Kruzhkov and Faminskii [11], Kato [9], Bourgain [3], Saut [21], Temam [23], Ponce and his colleages [10,17], etc. Last years appeared papers on initial boundary value problems for dispersive equations in bounded and non-bounded domains. Here we can mention again Bona and his colleagues [2], Bubnov [4], Faminskii [6], Faminskii and Larkin [7], Larkin [13,14].

Quite recently was discovered that the KdV equation has an implicit internal dissipation. This property allowed to prove exponential decay of small solutions in bounded domains without adding any artificial damping. Later, this effect was proved for a wide class of dispersive equations of any odd order in the space variable. We can mention here papers of Larkin [13,14], Faminskii and Larkin [7,8]. In [20] Rosier showed that control of the linear KdV equation with a "drift term", u_x, is impossible for the critical domains. It means that there is not decay of solutions with time for a set of critical domains. Hence, there is not also decay of solutions in a quarter-plane. By the way, without the "drift term" it is possible to prove exponential decay of small solutions for the KdV equation posed on any bounded interval $(0, L)$.

Recently appeared papers of Faminskii [6], Pyatkov [19], Linares and Pastor [15], Linares and Saut [18] on initial boundary value problems for the Zakharov-Kuznetsov (ZK) equation which may be considered as multi-dimensional analogue

2000 *Mathematics Subject Classification:* 35M20, 35Q72 of the KdV equation , see [24]. Our work was motivated by the paper of Saut and Temam [22] on initial boundary value problem in a domain bounded in x variable and non-bounded in y variable. Studying this paper, we discovered that the term u_{xyy} in ZK equation delivers additional "dissipation" which can help to prove decay of small solutions in non-bounded domains of a channel type non-bounded in x direction and we consider the following initial boundary value problem.

2. Formulation of the problem and main results

Let T, L be real positive numbers;

$$D = \left\{(x,y) \in \mathbb{R}^2 : \quad x>0, \quad y \in (0,L)\right\};$$
$$Q_t = D \times (0,t), \quad t \in (0,T).$$

Consider in Q_t the following initial boundary value problem:

$$Lu \equiv u_t + \alpha u_x + u u_x + \Delta u_x = 0 \text{ in } Q_t; \tag{2.1}$$
$$u(0,y,t) = u(x,0,t) = u(x,L,t) = 0,$$
$$y \in (0,L), \quad x>0, \quad t>0; \tag{2.2}$$
$$u(x,y,0) = u_0(x,y), \quad (x,y) \in D; \tag{2.3}$$

where $\alpha = 0$ or $\alpha = 1$, $\Delta = D_x^2 + D_y^2$. Exploiting regularization of (2.1)-(2.3) by a parabolic problem as in [22], one can prove the following result.

Theorem 2.1. *Let T and L be arbitrary finite real positive numbers. Let*

$$u_0 \in H^1(D), \quad u_{0y} \in H^1(D), \quad \Delta u_{0x} \in L^2(D),$$
$$u_0(0,y) = u_{0yy}(0,y) = u_0(x,0) = u_0(x,L) = 0,$$

and there is a real positive k such that

$$\int_D e^{kx} \left[\Delta u_{0x} + u_0 u_{0x} + u_{0x}\right]^2 dxdy < \infty.$$

Then there exists a unique regular solution of (2.1)-(2.3) such that

$$u \in L^\infty(0,T;H^2(D)),$$
$$\Delta u_x \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^1(D)),$$
$$u_x \in L^2(0,T;H^2(D)), u_t \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^1(D)).$$

3. Decay of Solutions

The main result of this article is the following theorem.

Theorem 3.1. *Let $\alpha = 1$, $L \in (0, 2\sqrt{2})$ and $\|u_0\|^2_{L^2(D)} \le \dfrac{9(8-L^2)^2}{800L^2}$. Then regular solutions of (2.1)-(2.3) satisfy the inequality*

$$\|u\|^2_{L^2(D)}(t) \le e^{-\chi t}(e^{kx}, u_0^2),$$

where $\chi = \dfrac{8\sqrt{2}(8-L^2)^{3/2}}{125L^3}$, $k = \dfrac{\sqrt{2(8-L^2)}}{\sqrt{5}L}$.

Proof: Transforming the integral

$$(u, Lu)(t) = (u, u_t)(t) + (u, u_x)(t) + (u^2, u_x)(t) + (u, \Delta u_x)(t) = 0 \tag{3.1}$$

to the equality

$$\|u\|^2(t) + \int_0^t \int_0^L u_x^2(0, y, \tau)\,dy\,d\tau = \|u_0\|^2,$$

we get

$$\|u\|^2(t) \le \|u_0\|^2, \quad t > 0. \tag{3.2}$$

Next, consider for some $k > 0$ the equality

$$\begin{aligned}&(e^{kx}u, Lu)(t) = (e^{kx}u, u_t)(t) + (e^{kx}u, u_x)(t)\\ &+(e^{kx}u^2, u_x)(t) + (e^{kx}u, \Delta u_x)(t) = 0\end{aligned}$$

which can be reduced to the form

$$\begin{aligned}&\frac{d}{dt}(e^{kx}, u^2)(t) + k(e^{kx}, u_y^2)(t) + 3k(e^{kx}, u_x^2)(t)\\ &+\int_0^L u_x^2(0, y, t)\,dy - (k + k^3)(e^{kx}, u^2)(t) - \frac{2k}{3}(e^{kx}, u^3)(t) = 0.\end{aligned} \tag{3.3}$$

It is easy to prove

Proposition 3.2. *Let* $\varphi \in H_0^1(0, L)$*, then*

$$\int_0^L \varphi^2(y)\,dy \le \frac{L^2}{8}\int_0^L \varphi_y^2(y)\,dy. \tag{3.4}$$

Proof: Let $y \in (0, \frac{L}{2})$, then

$$\varphi(y) = \int_0^y \varphi_s(s)\,ds \le y^{1/2}\Big(\int_0^y \varphi_s^2(s)\,ds\Big)^{1/2}$$

and

$$\int_0^{L/2} \varphi^2(y)\,dy \le \int_0^{L/2} y \int_0^y \varphi_s^2(s)\,ds\,dy \le \frac{L^2}{8}\int_0^{L/2} \varphi_y^2(y)\,dy.$$

Analogously,

$$\int_{L/2}^L \varphi^2(y)\,dy \le \frac{L^2}{8}\int_{L/2}^L \varphi_y^2(y)\,dy.$$

Finally,

$$\int_0^L \varphi^2(y)dy \le \frac{L^2}{8}\int_0^L \varphi_y^2(y)dy.$$

Proposition is proved. $\square$

It is known, [12], that

$$\|\varphi\|^2_{L^4(\mathbb{R}^2)} \le 2\|\varphi\|^2_{L^2(\mathbb{R}^2)}\|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)}. \tag{3.5}$$

Using (3.5), we calculate

$$\begin{aligned} I &= -\frac{2k}{3}(e^{kx},u^3)(t) \le \frac{2k}{3}\|u\|(t)\|e^{kx}u^2\|(t) \\ &\le \frac{2k}{3}\|u\|(t)\|e^{kx/2}u\|^2_{L^4(D)}(t) \\ &\le \frac{4k}{3}\|u\|(t)\|e^{kx/2}u\|(t)\|\nabla(e^{kx/2}u)\|(t). \end{aligned}$$

Taking into account (3.2), we continue

$$I \le \frac{4k}{3}\|u_0\|\|e^{kx/2}u\|(t)\left\{(e^{kx},[u_y^2+\frac{k^2}{2}u^2+2u_x^2])(t)\right\}^{1/2}.$$

Making use of the Young inequality, we get

$$\begin{aligned} I &\le 2\epsilon(e^{kx},u_y^2)(t)+4\epsilon(e^{kx},u_x^2)(t)+\epsilon k^2(e^{kx},u^2)(t) \\ &+ \frac{2k^2}{\epsilon}\|u_0\|^2(e^{kx},u^2)(t), \end{aligned} \tag{3.6}$$

where ϵ is an arbitrary positive number.
Substituting I into (3.3), we come to the inequality

$$\begin{aligned} &\frac{d}{dt}(e^{kx},u^2)(t)+(k-2\epsilon)(e^{kx},u_y^2)(t)+(3k-4\epsilon)(e^{kx},u_x^2)(t) \\ &-(k+\epsilon k+k^3)(e^{kx},u^2)(t)-\frac{2k^2}{\epsilon}\|u_0\|^2(e^{kx},u^2)(t)\le 0. \end{aligned} \tag{3.7}$$

Taking $0<\epsilon\le\frac{k}{4}$ and exploiting (3.4), we get

$$\begin{aligned} &\frac{d}{dt}(e^{kx},u^2)(t)+\left[k\left(\frac{8}{L^2}-1\right)-\frac{16\epsilon}{L^2}\right](e^{kx},u^2)(t) \\ &-\frac{5k^3}{4}(e^{kx},u^2)(t)-\frac{2k^2}{9\epsilon}\|u_0\|^2(e^{kx},u^2)(t)\le 0. \end{aligned} \tag{3.8}$$

Denote

$$\frac{8}{L^2}-1=4\delta^2>0, \tag{3.9}$$

whence

$$\delta = \frac{1}{2L}\sqrt{8 - L^2}. \tag{3.10}$$

Putting $\dfrac{16\epsilon}{L^2} = 2k\delta^2$, (3.7) reads

$$\frac{d}{dt}(e^{kx}, u^2)(t) + 2k\left(\delta^2 - \frac{5}{8}k^2\right)(e^{kx}, u^2)(t) - \frac{2k^2}{9\epsilon}\|u_0\|^2(e^{kx}, u^2)(t) \le 0. \tag{3.11}$$

Now we choose $k > 0$ such that $\delta^2 - \dfrac{5}{8}k^2 > 0$. For this purpose, put $k^2 = \dfrac{8}{5}\gamma^2\delta^2$, or $k = \sqrt{\dfrac{8}{5}}\gamma\delta$, where $\gamma \in (0, 1)$.
With this choice of $k > 0$, (3.11) becomes

$$\frac{d}{dt}(e^{kx}, u^2)(t) + 2k\left[(1-\gamma^2)\delta^2 - \frac{k}{2\epsilon}\|u_0\|^2\right](e^{kx}, u^2)(t) \le 0.$$

Now we assume $\|u_0\|^2$ such that

$$\frac{k}{2\epsilon}\|u_0\|^2 \le \gamma^2(1-\gamma^2)\delta^2 \tag{3.12}$$

which gives

$$\frac{d}{dt}(e^{kx}, u^2)(t) + \chi(e^{kx}, u^2)(t) \le 0, \tag{3.13}$$

where

$$\begin{aligned}
\chi &= 2k(1-\gamma^2)^2\frac{(8-L^2)}{4L^2} = \sqrt{\frac{8}{5}}\frac{\gamma(1-\gamma^2)^2}{4L^3}(8-L^2)^{3/2} \\
&= \frac{1}{2}\sqrt{\frac{2}{5}}\frac{\gamma(1-\gamma^2)^2}{L^3}(8-L^2)^{3/2}.
\end{aligned}$$

The function $A(\gamma) = \gamma(1-\gamma^2)^2$ has its maximal value when $\gamma^2 = \dfrac{1}{5}$, hence

$$\max_{\gamma>0,\gamma\neq 1} A(\gamma) = \frac{16}{25\sqrt{5}}.$$

With this, χ becomes

$$\chi = \frac{8\sqrt{2}}{125}\frac{(8-L^2)^{3/2}}{L^3}.$$

Solving (3.13), we prove Theorem. $\square$

Theorem 3.3. *Let $\alpha = 0$, $L > 0$, $k = \frac{4\sqrt{3}}{5}L$, $\|u_0\| \leq \frac{\sqrt{3}}{5L}$. Then regular solutions of (2.1)-(2.3) satisfy the following inequality*

$$(e^{kx}, u^2)(t) \leq e^{-\chi t}(e^{kx}, u_0^2),$$

where $\chi = \frac{96\sqrt{3}}{125L}$.

Proof: Putting $\alpha = 0$ in (2.1), multiplying it by $e^{kx}u$, and taking into account (3.6), we come to the inequality

$$\begin{aligned}&\frac{d}{dt}(e^{kx}, u^2)(t) + (k - 2\epsilon)(e^{kx}, u_y^2)(t) - k^2(\epsilon + k)(e^{kx}, u^2)(t)\\ &+(3k - 4\epsilon)(e^{kx}, u_x^2)(t) - \frac{2k^2}{\epsilon}\|u_0\|^2(e^{kx}, u^2)(t) \leq 0.\end{aligned} \tag{3.14}$$

Putting $\epsilon = \frac{k}{4}$ and using (3.4), we get

$$\begin{aligned}&\frac{d}{dt}(e^{kx}, u^2)(t) + 4k\left(\frac{1}{L^2} - \frac{5}{16}k^2\right)(e^{kx}, u^2)(t)\\ &-\frac{2k^2}{\epsilon}\|u_0\|^2(e^{kx}, u^2)(t) \leq 0.\end{aligned} \tag{3.15}$$

For $k = \frac{4\gamma}{\sqrt{5}}L$, where $\gamma \in (0, 1)$, (3.15) reads

$$\frac{d}{dt}(e^{kx}, u^2)(t) + 4k\left[\frac{(1 - \gamma^2)}{L^2} - 2\|u_0\|^2\right](e^{kx}, u^2)(t) \leq 0.$$

Taking $\|u_0\|^2 = \frac{\gamma^2(1 - \gamma^2)}{2L^2}$, we find

$$\frac{d}{dt}(e^{kx}, u^2)(t) + \chi(e^{kx}, u^2)(t) \leq 0, \tag{3.16}$$

where $\chi = \frac{16}{\sqrt{5}}\frac{\gamma^3(1 - \gamma^2)}{L}$.

The function $A(\gamma) = \gamma^3(1 - \gamma^2)$ has its maximal value when $\gamma = \sqrt{\frac{3}{5}}$, which gives

$$\chi = \frac{96\sqrt{3}}{125L}, \quad k = \frac{4\sqrt{3}}{5}L, \quad \|u_0\| \leq \frac{\sqrt{3}}{5}L.$$

Solving (3.16), we complete the proof of Theorem. $\square$

Remark. The presence in (2.1) of a linear term $u_x(\alpha = 1)$ implies a restriction for value of $L : (L < 2\sqrt{2})$, which means that a channel D has a limitation in width. On the other hand , absence of that term $(\alpha = 0)$ allows to L bo be any finite positive number; it means that a channel may be of any finite width.

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