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Global behavior of the difference equation $x_{n+1} = \frac{Ax_{n-1}}{B-Cx_nx_{n-2}}$

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ABSTRACT: The aim of this work is to investigate the global stability, periodic nature, oscillation and the boundedness of all admissible solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}}, \qquad n = 0, 1, 2, \ldots$$

where A, B, C are positive real numbers.

Key Words: difference equation, periodic solution, globally asymptotically stable.

Contents

1. Introduction and Preliminaries

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and non-rational difference equations, one can refer to the monographs [1,3,4,5,6] and references therein.

M. Aloqeili in [2] discussed the stability properties and semi-cycle behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_nx_{n-1}}, \qquad n = 0, 1, \ldots$$

with real initial conditions and positive real number a.
In this paper, we study the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}}, \qquad n = 0, 1, \ldots \tag{1}$$

2000 *Mathematics Subject Classification:* 39A11 where A, B, C are nonnegative real numbers.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \qquad n = 0, 1, \cdots \tag{2}$$

where $f : R^{k+1} \to R$.

Definition 1.1 *[4].*

An equilibrium point for equation (2) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$.

Definition 1.2 *[4].*

1. *An equilibrium point $\bar{x}$ for equation (2) is called locally stable if for every $\epsilon > 0$, $\exists \delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$ is such that $x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[, \forall n \in N$. Otherwise $\bar{x}$ is said to be unstable.*

2. *The equilibrium point $\bar{x}$ of equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.*

3. *An equilibrium point $\bar{x}$ for equation (2) is called global attractor if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.*

4. *The equilibrium point $\bar{x}$ for equation (2) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.*

The linearized equation associated with equation (2) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\bar{x}, \dots, \bar{x}) y_{n-i}, \qquad n = 0, 1, 2, \dots \tag{3}$$

The characteristic equation associated with equation (3) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\bar{x}, \dots, \bar{x}) \lambda^{k-i} = 0. \tag{4}$$

Theorem 1.3 *[4]. Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation (2) . Then the following statements are true:*

1. *If all roots of equation (4) lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*

2. *If at least one root of equation (4) has absolute value greater than one, then $\bar{x}$ is unstable.*

The change of variables $x_n = \sqrt{\frac{B}{C}} y_n$ reduces equation (1) to the difference equation

$$y_{n+1} = \frac{r y_{n-1}}{1 - y_n y_{n-2}}, \qquad n = 0, 1, 2, \dots \tag{5}$$

where $r = \frac{A}{B}$.

2. Linearized stability analysis

In this section we study the local asymptotic stability of the equilibrium points of equation (5). We can see that equation (5) has the equilibrium points $\bar{y} = 0$ and $\bar{y_1} = \sqrt{1-r}$, $\bar{y_2} = -\sqrt{1-r}$ when $r < 1$ and the zero equilibrium only when $r \geq 1$. The linearized equation associated with equation (5) about $\bar{y}$ is

$$z_{n+1} - \frac{r}{1-\bar{y}^2} z_{n-1} - \frac{r\bar{y}^2}{(1-\bar{y}^2)^2}(z_n + z_{n-2}) = 0, \qquad n = 0, 1, 2, ... \tag{6}$$

The characteristic equation associated with this equation is

$$\lambda^3 - \frac{r}{1-\bar{y}^2}\lambda - \frac{r\bar{y}^2}{(1-\bar{y}^2)^2}(\lambda^2 + 1) = 0. \tag{7}$$

We summarize the results of this section in the following theorem.

Theorem 2.1 *1. If $r > 1$, then the zero equilibrium point is a saddle point.*

2. If $r < 1$, then the equilibrium points $\bar{y} = 0$ is locally asymptotically stable and $\bar{y_1} = \sqrt{1-r}$, $\bar{y_2} = -\sqrt{1-r}$ are unstable.

Proof: *The linearized equation associated with equation (5) about $\bar{y} = 0$ is*

$$z_{n+1} - rz_{n-1} = 0, \qquad n = 0, 1, 2, ...$$

The characteristic equation associated with this equation is

$$\lambda^3 - r\lambda = 0.$$

That is $\lambda = 0, \pm\sqrt{r}$.

1. *If $r < 1$, then $|\lambda| < 1$ for all roots and $\bar{y} = 0$ is locally asymptotically stable.*

2. *If $r > 1$, it follows that $\bar{y} = 0$ is unstable (saddle point).*
   The linearized equation (6) about $\bar{y} = \pm\sqrt{1-r}$ becomes
   $z_{n+1} - z_{n-1} - (\frac{1}{r})(1-r)(z_n + z_{n-2}) = 0, \qquad n = 0, 1, 2, \ldots$
   The associated characteristic equation is

   $\lambda^3 - \lambda - (\frac{1}{r})(1-r)(\lambda^2 + 1) = 0.$
   Let $f(\lambda) = \lambda^3 - \lambda - (\frac{1}{r})(1-r)(\lambda^2 + 1)$.
   We can see that $f(\lambda)$ has a root in $(1, \infty)$. Then $\bar{y_1} = \sqrt{1-r}$, $\bar{y_2} = -\sqrt{1-r}$ are unstable.

□

3. Oscillation

Theorem 3.1 *Assume that $r < 1$. Then the interval $(-\sqrt{1-r}, \sqrt{1-r})$ is an invariant interval for equation (5).*

Proof: *The proof is by induction. Suppose that $y_{-2}, y_{-1}, y_0 \in (-\sqrt{1-r}, \sqrt{1-r})$. Hence $| y_{-i} |< \sqrt{1-r}$, $i = 0, 1, 2$. This implies that $| y_{-2}y_0 |< 1-r$.*
Then

$$| y_1 |= \frac{r \mid y_{-1} \mid}{\mid 1 - y_0 y_{-2} \mid} < \frac{r \mid y_{-1} \mid}{\mid 1 - \mid y_0 y_{-2} \mid\mid} <| y_{-1} |,$$

$$| y_2 |= \frac{r \mid y_0 \mid}{\mid 1 - y_1 y_{-1} \mid} < \frac{r \mid y_0 \mid}{\mid 1 - \mid y_1 y_{-1} \mid\mid} <| y_0 |,$$

where $| y_1 |<| y_{-1} |< \sqrt{1-r}$.
Now if for a certain $n_0 \in \mathbb{N}$ we have $y_{n_0-2}, y_{n_0-1}, y_{n_0} \in (-\sqrt{1-r}, \sqrt{1-r})$, then $| y_{n_0+1} |= \frac{r|y_{n_0-1}|}{|1-y_{n_0}y_{n_0-2}|} < \frac{r|y_{n_0-1}|}{|1-|y_{n_0}y_{n_0-2}||} <| y_{n_0-1} |< \sqrt{1-r}$. This completes the proof.

□

Corollary 3.2 *Assume that $\{y_n\}_{n=-2}^{\infty}$ be a solution of equation (5) such that either $y_{-2}, y_{-1}, y_0 \in (0, \sqrt{1-r})$ (or $(-\sqrt{1-r}, 0)$). Then $\{y_n\}_{n=-2}^{\infty}$ is positive (or negative). Moreover, $\{y_n\}_{n=-2}^{\infty}$ decreases (or increases) to the zero equilibrium point.*

Theorem 3.3 *Let $\{y_n\}_{n=-2}^{\infty}$ be a nontrivial solution of equation (5) and let $\bar{y_1} = \sqrt{1-r}$, $\bar{y_2} = -\sqrt{1-r}$ denote the nonzero equilibrium points of equation (5) such that either,*
(C_1) $\bar{y_2} = -\sqrt{1-r} < y_{-1} < 0 < y_{-2}, y_0 < \sqrt{1-r} = \bar{y_1}$
or
(C_2) $\bar{y_2} = -\sqrt{1-r} < y_{-2}, y_0 < 0 < y_{-1} < \sqrt{1-r} = \bar{y_1}$
is satisfied. Then $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y} = 0$ with semicycles of length one. Moreover $y_{2n+2} < (>) y_{2n}$ and $y_{2n+1} > (<) y_{2n-1}$, $n = 0, 1, 2, \ldots$.

Proof: *Assume that condition (C_1) is satisfied. Then we have*
$y_1 = \frac{ry_{-1}}{1-y_0y_{-2}} > y_{-1}$, $y_2 = \frac{ry_0}{r-y_{-1}y_1} < y_0$.
Now suppose that for a fixed $n_0 \in \mathbb{N}$ we have

$$-\sqrt{1-r} < y_{2n_0-1} < y_{2n_0+1} < 0 \quad \text{and} \quad 0 < y_{2n_0} < y_{2n_0-2} < \sqrt{1-r}.$$

Then

$$0 > y_{2n_0+3} = \frac{ry_{2n_0+1}}{1 - y_{2n_0+2}y_{2n_0}} > y_{2n_0+1},$$

and

$$0 < y_{2n_0+2} = \frac{ry_{2n_0}}{1 - y_{2n_0+1}y_{2n_0-1}} < y_{2n_0}.$$

Therefore, $y_{2n} > y_{2n+2} > 0$ and $y_{2n-1} < y_{2n+1} < 0$ for all $n \geq -1$ and the result follows.
For condition (C_2), the result is similar and will be omitted.

□

4. Global behavior of equation (5)

Theorem 4.1 *The following statements are true.*

1. *If $r < 1$, then the zero equilibrium point is a global attractor with basin $(-\sqrt{1-r}, \sqrt{1-r})^3$.*
2. *If $r = 1$, then equation (5) has prime period two solutions of the form $\ldots, 0, \varphi, 0, \varphi, 0, \ldots$, where $\varphi \in \mathbb{R}$.*
3. *If $r > 1$, then there exist solutions which are neither bounded nor persist.*

Proof:

1. *Suppose that $y_{-2}, y_{-1}, y_0 \in (-\sqrt{1-r}, \sqrt{1-r})$. Then using theorem (3.1) we have that $y_n \in (-\sqrt{1-r}, \sqrt{1-r})$, $n \geq 1$.*
Moreover, we have $\mid y_{n+1} \mid < \mid y_{n-1} \mid$, $n = 0, 1, \ldots.$
That is $\mid y_{2n+1} \mid < \mid y_{2n-1} \mid$ and $\mid y_{2n+2} \mid < \mid y_{2n} \mid$, $n = 0, 1, \ldots$
From equation (5) we have
$$\mid y_{2n+1} \mid = \frac{r \mid y_{2n-1} \mid}{\mid 1 - y_{2n}y_{2n-2} \mid} \leq \frac{r \mid y_{2n-1} \mid}{\mid 1 - \mid y_{2n}y_{2n-2} \mid\mid}$$
and
$$\mid y_{2n+2} \mid = \frac{r \mid y_{2n} \mid}{\mid 1 - y_{2n+1}y_{2n-1} \mid} \leq \frac{r \mid y_{2n} \mid}{\mid 1 - \mid y_{2n+1}y_{2n-1} \mid\mid}.$$
Now suppose that $\lim_{n\to\infty} \mid y_{2n+1} \mid = L$ and $\lim_{n\to\infty} \mid y_{2n} \mid = M$. Then
$$L \leq \frac{rL}{\mid 1 - M^2 \mid} \quad \text{and} \quad M \leq \frac{rM}{\mid 1 - L^2 \mid}.$$
If $L \neq 0$, then $\mid 1 - M^2 \mid \leq r$. This implies that $\sqrt{1-r} \leq M \leq \sqrt{1+r}$, which is a contradiction. Hence we have $L = 0$. The second inequality gives $M \leq rM$, from which $M = 0$ where $r < 1$. Therefore, $\{y_n\}_{n=-2}^{\infty}$ converges to zero. This completes the proof.

2. *Clear!*

3. *Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of equation (5) with the initial conditions, $\mid y_{-i} \mid < \sqrt{r-1} (> \sqrt{r+1})$, $i = 0, 2$ and $\mid y_{-i} \mid > \sqrt{r+1} (< \sqrt{r-1})$, $i = 1$. We consider only the case $\mid y_{-i} \mid < \sqrt{r-1}$, $i = 0, 2$ and $\mid y_{-i} \mid > \sqrt{r+1}$, $i = 1$. It follows that $\mid y_{-2}y_0 \mid = \mid y_{-2} \mid\mid y_0 \mid < r - 1$. That is $-r + 1 < y_{-2}y_0 < r - 1$. This implies that $-r + 2 < 1 - y_{-2}y_0 < r$. Hence we have*
$$\mid y_1 \mid = \frac{\mid ry_{-1} \mid}{\mid 1 - y_0y_{-2} \mid} > \frac{r \mid y_{-1} \mid}{r} = \mid y_{-1} \mid > \sqrt{r+1}.$$
It follows that $\mid y_1y_{-1} \mid = \mid y_1 \mid\mid y_{-1} \mid > r + 1$, which implies that $r + 2 < 1 - y_1y_{-1} < -r$ and so
$$\mid y_2 \mid = \frac{\mid ry_0 \mid}{\mid 1 - y_1y_{-1} \mid} < \frac{r \mid y_0 \mid}{r} = \mid y_0 \mid < \sqrt{r-1}.$$

By induction we get $|\ y_{2n+1}\ |>|\ y_{2n-1}\ |>\sqrt{r+1}$ *and* $|\ y_{2n+2}\ |<|\ y_{2n}\ |<\sqrt{r-1}$, $n\geq 0$.

Now suppose that $|\ y_{2n}\ |\to L$ *and* $|\ y_{2n+1}\ |\to M$ *as* $n\to\infty$. *But*

$$|\ y_{2n+2}\ |=\frac{r\ |\ y_{2n}\ |}{|\ 1-y_{2n+1}y_{2n-1}\ |}\leq\frac{r\ |\ y_{2n}\ |}{|\ 1-|\ y_{2n+1}y_{2n-1}\ ||}.$$

Then $L\leq\frac{rL}{|1-M^2|}$. *If* $L\neq 0$, *then* $|\ 1-M^2\ |\leq r$. *This implies that* $M\leq\sqrt{r+1}$. *This is a contradiction and so* $L=0$. *Now, as*

$$|\ y_{2n+1}\ |=\frac{r\ |\ y_{2n-1}\ |}{|\ 1-y_{2n}y_{2n-2}\ |}\geq\frac{r\ |\ y_{2n-1}\ |}{1+|\ y_{2n}\ ||\ y_{2n-2}\ |},$$

then we have $M\geq\frac{rM}{1+L^2}=rM$ *and therefore,* $M=\infty$.
The case when $|\ y_{-i}\ |>\sqrt{r+1}$, $i=0,2$ *and* $|\ y_{-i}\ |<\sqrt{r-1}$, $i=1$ *is similar and will be omitted.*

□

Conjecture Assume that $r<1$. Then the zero equilibrium point is global asymptotically stable (in the set of all admissible solutions).

5. Numerical examples

Example 5.1 *Figure* 1. *shows that if* $r=0.6$, *then the solution* $\{y_n\}_{n=-2}^{\infty}$ *with initial conditions* $y_{-2}=0.3$, $y_{-1}=-0.2$, $y_0=0.35$ *converges to zero.*

Example 5.2 *Figure* 2. *shows that if* $r=1.9$, *then the solution* $\{y_n\}_{n=-2}^{\infty}$ *with initial conditions* $y_{-2}=0.9$, $y_{-1}=-1.7$, $y_0=0.9$ *is neither bounded nor persist.*

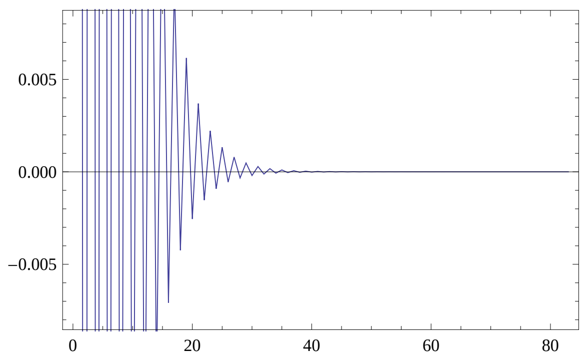

Figure 1: The difference equation $y_{n+1}=\frac{0.6y_{n-1}}{1-y_{n-2}y_n}$

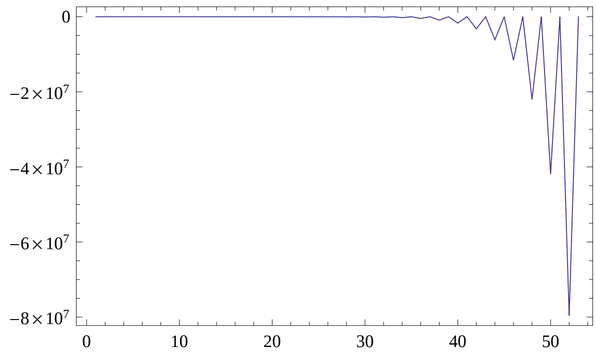

Figure 2: The difference equation $y_{n+1} = \frac{1.9y_{n-1}}{1-y_{n-2}y_n}$

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