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Some separation axioms in generalized topological spaces*

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abstract: We give different definitions for g-closed sets, R_0 and R_1 spaces in generalized topological spaces, characterize such spaces and compare with the existing definitions and results.

Key Words: generalized topology, $\mu-$closed and $\mu-$open sets; $\delta-$open and $\delta-$closed sets, connected, irreducible, $\mu-$regular, R_0 and R_1 generalized topological spaces.

Contents

1. Introduction and preliminaries

A *generalized topology* or simply GT μ [3] on a nonempty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary union. Elements of μ are called $\mu-$*open* sets. A subset A of X is said to be $\mu-$*closed* if $X - A$ is $\mu-$open. The pair (X, μ) is called a *generalized topological space* (GTS). If A is a subset of a space (X, μ), then $c_\mu(A)$ is the smallest $\mu-$closed set containing A and $i_\mu(A)$ is the largest $\mu-$open set contained in A. If $\gamma : \wp(X) \to \wp(X)$ is a monotonic function defined on a nonempty set X and $\mu = \{A \mid A \subset \gamma(A)\}$, the family of all $\gamma-$open sets is also a GT [2], $i_\mu = i_\gamma$, $c_\mu = c_\gamma$ and $\mu = \{A \mid A = i_\mu(A)\}$ [4, Corollary 1.3]. The family of all monotonic functions defined on X is denoted by Γ. By a space (X, μ), we will always mean a GTS (X, μ). A subset A of a space (X, μ) is said to be $\alpha-$*open* [4] (resp., *semiopen* [4], *preopen* [4], $b-$*open* [14], $\beta-$*open* [4]) if $A \subset i_\mu c_\mu i_\mu(A)$ (resp.,$A \subset c_\mu i_\mu(A), A \subset i_\mu c_\mu(A), A \subset i_\mu c_\mu(A) \cup c_\mu i_\mu(A), A \subset c_\mu i_\mu c_\mu(A)$). We will denote the family of all $\alpha-$open sets by α, the family of all semiopen sets by σ, the family of all preopen sets by π, the family of all $b-$open sets by b and the family of all $\beta-$open sets by β. If

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(X,μ) is a GTS, then we say that a subset $A \in \delta \subset \wp(X)$ [6] if for every $x \in A$, there exists a $\mu-$closed set Q such that $x \in i_\mu(Q) \subset A$. Then (X,δ) is a GTS [6, Proposition 2.1] such that $\delta \subset \mu$ [6, Theorem 1]. Elements of δ are called the $\delta-$open sets of (X,μ). For $A \subset X$, $i_\delta(A)$ and $c_\delta(A)$ are the *interior* and *closure* of A in (X,δ). We will denote by ν (resp. ξ, η, ε, ψ), the family of all $\alpha-$open (resp. semiopen, preopen, $b-$open, $\beta-$open) sets of the generalized space (X,δ). If $\kappa \in \{\mu,\ \alpha,\ \sigma,\ \pi,\ b,\ \beta,\ \delta,\ \nu,\ \xi,\ \eta,\ \varepsilon,\ \psi\}$ and A is a subset of a space (X,κ), then $c_\kappa(A)$ is the smallest $\kappa-$closed set containing A and $i_\kappa(A)$ is the largest $\kappa-$open set contained in A. Note that the operator c_κ is monotonic, increasing and idempotent and the operator i_κ is monotonic, decreasing and idempotent. Clearly, A is $\kappa-$open if and only if $A = i_\kappa(A)$ and A is $\kappa-$closed if and only if $A = c_\kappa(A)$. Also, for every subset A of a space (X,κ), $X - i_\kappa(A) = c_\kappa(X - A)$. If $\lambda \subset \wp(X)$ is a GT, then $\gamma \in \Gamma$ is said to be $\lambda - friendly$ [5] if $\gamma(A) \cap L \subset \gamma(A \cap L)$ for $A \subset X$ and $L \in \lambda$. In [14], it is denoted that $\Gamma_4 = \{\gamma \mid \gamma$ is $\mu-$friendly where μ is the GT of all $\gamma-$open sets$\}$ and if $\gamma \in \Gamma_4$, the space (X,γ)(resp. (X,μ)) is called a $\gamma-$space. By [14, Theorem 2.1], the intersection of two $\mu-$open sets is again a $\mu-$open set and so every $\gamma-$space is a quasi-topological space [5]. By [14, Theorem 2.3], it is established that in a $\gamma-$space, i_μ and c_μ preserve finite intersection and finite union, respectively. Later, in [5], it is established that the above result is also true for quasi-topological spaces. A space (X,μ) is said to be *strong* if $X \in \mu$. The following lemma is essential to proceed further where the easy proof is omitted.

Lemma 1.1. *Let (X,μ) be a space where μ is the family of all $\gamma-$open sets of a $\gamma \in \Gamma_4$. Then the following hold.*
(a) The intersection of two $\delta-$open sets is a $\delta-$open set.
(b) $i_\delta(A) \cap i_\delta(B) = i_\delta(A \cap B)$ for every subsets A and B of X.
(c) $c_\delta(A) \cup c_\delta(B) = c_\delta(A \cup B)$ for every subsets A and B of X.
(d) $i_\delta \in \Gamma_4$.

2. Strong generalized spaces

If (X,μ) is any generalized space which is not strong, then in [7, Proposition 1.2], it is established that $X \in \sigma$ and so it follows that always $X \in b$ and $X \in \beta$. The following Example 2.1 shows that in general, if $X \notin \mu$, then $X \notin \lambda$ for $\lambda \in \{\mu,\ \delta,\ \alpha,\ \pi,\ \nu,\ \eta\}$ and Theorem 2.1 below shows that $X \in \xi$ and hence $X \in \varepsilon$ and $X \in \psi$.

Example 2.1. *Let X be the set of all real numbers and $\mu = \{\emptyset, \{0\}\}$. Then $X \notin \lambda$ where $\lambda \in \{\mu,\ \delta,\ \alpha,\ \pi,\ \nu,\ \eta\}$.*

Theorem 2.1. *If (X,μ) is a generalized space which is not strong, then the following hold.*
(a) $X \notin \pi$ and hence $X \notin \alpha$.
(b) $X \notin \delta$ and hence $X \notin \eta$ and $X \notin \nu$.
(c)$X \in \xi$ and hence $X \in \varepsilon$ and $X \in \psi$.

Proof: (a) Suppose $X \in \pi$. But always, $X \in \sigma$ and so $X \in \sigma \cap \pi = \alpha$. Therefore, $X \subset i_\mu c_\mu i_\mu(X) \subset i_\mu c_\mu(X) = i_\mu(X)$. Hence $X \in \mu$, a contradiction and so $X \notin \pi$

and hence $X \notin \alpha$.
(b) Since $X \notin \mu$, $X \notin \delta$, since $\delta \subset \mu$. Since $\eta = \pi(\delta)$, by (a), $X \notin \eta$ and hence $X \notin \nu$, since $\nu = \alpha(\delta)$.
(c) Since $\xi = \sigma(\delta)$, $X \in \xi$ and so $X \in \varepsilon$ and $X \in \psi$. □

3. $g^{\star}_{\lambda}-$closed sets

Let (X, μ) be a generalized space. A subset A of X is said to be $g_{\mu}-$*closed* [9] if $c_{\mu}(A) \subset M$ whenever $A \subset M$ and $M \in \mu$. Various properties of $g_{\mu}-$closed are discussed and characterizations are given in [9] and these properties are valid for the generalized topologies induced by μ and δ. Given a topological space (X, τ) and a generalized topology μ on X, a subset A of X is said to be $g\mu-$*closed* [11] if $c_{\mu}(A) \subset M$ whenever $A \subset M$ and $M \in \tau$. *If $\mu = \tau$, then the $g\mu-$closed sets coincide with the $g-$closed sets of Levine [8]. If τ is fixed and μ is any one of the generalized topology, namely $\alpha, \sigma, \ \pi, \ b$ and β of the topological space (X, τ), where all these family contains X, then we have $g\alpha-$closed, gsemi$-$closed, gpre$-$closed, $gb-$closed and $g\beta-$closed sets in (X, τ) and all the results established in [11] are valid for these sets. If μ is a fixed generalized topology, and instead of τ, if we consider σ, b and β, the generalized topologies induced by μ, which contains X, then we can define $g\sigma(\mu)-$closed, $gb(\mu)-$closed and $g\beta(\mu)-$closed sets in the space (X, μ) and for these family of sets also, all the results established in [11] are valid.*

The difference between the two definitions is that the definition of $g\mu-$closed sets uses elements of the topology τ on X where $X \in \tau$ where as the definition of $g_{\mu}-$closed sets uses elements of the generalized topology μ where X may or may not be in μ. Therefore, the definition of $g_{\mu}-$closed sets is more general, since the definition uses a large class of generalized topologies which also contains the class of all topological spaces. Moreover, similar results established for $g\mu-$closed sets in [11] are already established for $g_{\mu}-$closed sets in [9]. We give below a new definition for generalized closed sets in a generalized space, which is common for both strong spaces and non-strong spaces, and discuss the relation between these three kinds of sets in the following Examples 3.1 to 3.3. A subset A of $\mathcal{M}_{\mu} = \cup\{B \mid B \in \mu\}$ of a generalized space (X, μ) is said to be $g^{\star}_{\mu}-$*closed* if $c_{\mu}(A) \cap \mathcal{M}_{\mu} \subset M$ whenever $A \subset M$ and $M \in \mu$. Note that, if the space is strong, then this definition coincides with the definition of $g_{\mu}-$*closed* sets.

Example 3.1. *Let X be a nonempty set and μ be a generalized topology on X. Suppose $\mathcal{M}_{\mu} = \cup\{A \mid A \in \mu\} \neq X$ and $\tau = \wp(\mathcal{M}_{\mu}) \cup \{X\}$. Then every $\mu-$closed subset of X contains $X - \mathcal{M}_{\mu}$. Therefore, every subset A of $\mathcal{M}_{\mu}$ is neither a $g_{\mu}-$closed set nor a $g\mu-$closed set. $g^{\star}_{\mu}-$closed sets depend on the generalized topology μ. Every nonempty subset B of X such that $B \cap (X - \mathcal{M}_{\mu}) \neq \emptyset$ or $B \subset (X - \mathcal{M}_{\mu})$ is not contained in any $\mu-$open set which implies that such sets are trivially $g_{\mu}-$closed. Clearly, such sets are $g\mu-$closed, since X is the only open set containing such sets.*

Example 3.2. *[1, Example 2.1] Let $X = \mathcal{I}_n = \{1, 2, 3, ..., n\}$. Define $\kappa : \wp(\mathcal{I}_n) \rightarrow \wp(\mathcal{I}_n)$ by $\kappa(A) = A$ if $\mathcal{I}_n - \{i\} \subseteq A$ for some $i \in \{1, 2, 3, ..., n\}$ and $\kappa(A) = \emptyset$*

otherwise. Then $\mu = \{\emptyset,\ X\} \cup \{A \subset \mathcal{I}_n \mid A = \mathcal{I}_n - \{i\}, i = 1, 2, 3, ...n\}$, *the co-singleton generalized topology defined on a finite set. The only* $\mu-$*closed sets are* $\emptyset$, X *and singleton subsets of* $\mathcal{I}_n$. *In this space, the family of all* $g^\star_\mu-$*closed sets, the family of all* $g_\mu-$*closed sets and family of all* $\mu-$*closed sets coincide. For the topology* $\tau = \{\emptyset\} \cup \{G \subset X \mid \{1, 2\} \subset G\}$ *on* X*, the* $\mu-$*closed sets are precisely the* $g\mu-$*closed sets.*

Example 3.3. *Consider the space* (X, τ) *and generalized topology* μ *of the Example 2.3 of [11]. In this space,* $\{a, c\}$ *is* $g\mu-$*closed but it is not* $g^\star_\mu-$*closed and also not* $g_\mu-$*closed.*

Throughout the paper, if μ *is a generalized topology on* X*, let* $\mathcal{M}_\mu = \cup\{A \mid A \in \mu\}$, $X \notin \mu$ and $\lambda \in \{\mu,\ \alpha,\ \pi,\ \sigma,\ b,\ \beta,\ \delta,\ \nu,\ \xi,\ \eta,\ \varepsilon,\ \psi\}$. Then, by Theorem 2.1, we have $\mathcal{M}_\lambda \neq X$ if $\lambda \in \{\mu,\ \alpha,\ \pi,\ \delta,\ \nu, \eta\}$ and $\mathcal{M}_\lambda = X$ if $\lambda \in \{\sigma,\ b,\ \beta,\ \xi,\ \varepsilon,\ \psi\}$. Moreover, $\mathcal{M}_\lambda = \mathcal{M}_\mu$, if $\mathcal{M}_\lambda \neq X$. The following Lemma 3.1 is essential to proceed further.

Lemma 3.1. *Let* X *be a nonempty set,* μ *be a generalized topology on* X *and* $A \subset X$. *Then the following hold.*
(a) $(X - \mathcal{M}_\lambda)$ *is a* $\lambda-$*closed set contained in every* $\lambda-$*closed set.*
(b) $c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda = c_\lambda(A) \cap \mathcal{M}_\lambda$.
(c) If A *is* $\lambda-$*closed, then* $c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda = A \cap \mathcal{M}_\lambda$.
(d) $c_\lambda(A) = (c_\lambda(A) \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda)$.
(e) If A *is* $\lambda-$*closed, then* $A = (A \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda)$.
(f) $(\mathcal{M}_\lambda, \lambda^\star)$ *is a strong generalized space where* $\lambda^\star = \lambda \mid \mathcal{M}_\lambda$ *is the subspace generalized topology.*
(g) If $A \subset \mathcal{M}_\lambda$*, then* $c_{\lambda*}(A) = c_\lambda(A) \cap \mathcal{M}_\lambda$ *and* $i_{\lambda\star}(A) = i_\lambda(A)$ *where* $c_{\lambda*}(A)$*(resp.* $i_{\lambda*}(A)$*) is the closure (resp. interior) of* A *in* $\mathcal{M}_\lambda$.
(h) $A \subset \mathcal{M}_\lambda$ *is* $\lambda^\star-$*closed in* $\mathcal{M}_\lambda$ *if and only if* $A = c_\lambda(A) \cap \mathcal{M}_\lambda$.
(i) $A \subset \mathcal{M}_\lambda$ *is* $\lambda^\star-$*closed in* $\mathcal{M}_\lambda$ *if and only if* $c_\lambda(A) = A \cup (X - \mathcal{M}_\lambda)$.

Proof: (a) follows from the fact that if G is $\lambda-$open, then $G \subset \mathcal{M}_\lambda$.
(b) Clearly, $c_\lambda(A\cap\mathcal{M}_\lambda)\cap\mathcal{M}_\lambda \subset c_\lambda(A)\cap\mathcal{M}_\lambda$. Let $x \in c_\lambda(A)\cap\mathcal{M}_\lambda$. Then $x \in c_\lambda(A)$ and $x \in \mathcal{M}_\lambda$. Now $x \in c_\lambda(A)$ implies that $G \cap A \neq \emptyset$ for every $\lambda-$open set G containing x and so $G \cap (A \cap \mathcal{M}_\lambda) \neq \emptyset$ for every $\lambda-$open set G containing x. Therefore, $x \in c_\lambda(A\cap\mathcal{M}_\lambda)$ and so $x \in c_\lambda(A\cap\mathcal{M}_\lambda)\cap\mathcal{M}_\lambda$. Hence $c_\lambda(A)\cap\mathcal{M}_\lambda \subset c_\lambda(A\cap\mathcal{M}_\lambda)\cap\mathcal{M}_\lambda$. This completes the proof.
(c) The proof follows from (b).
(d) $c_\lambda(A) = c_\lambda(A) \cap X = c_\lambda(A) \cap (\mathcal{M}_\lambda \cup (X - \mathcal{M}_\lambda)) = (c_\lambda(A) \cap \mathcal{M}_\lambda) \cup (c_\lambda(A) \cap (X - \mathcal{M}_\lambda)) = (c_\lambda(A) \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda)$, by (a).
(e) If A is $\lambda-$closed, by (d), we have $A = (A \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda)$.
The proofs of (f), (g), (h) and (i) are clear. □

As per the present definition, the $g^\star_\lambda-$closed sets must be subsets of $\mathcal{M}_\lambda$. Moreover, $g^\star_\lambda-$closed subsets coincide with $g_\lambda-$closed subsets if X is $\mu-$open. In Example 3.2, the space is strong and the $g^\star_\lambda-$closed sets are exactly the $g_\lambda-$closed sets.

It is easy to note that $g^\star_\lambda-$closed subsets are $g^\star_\lambda-$closed subsets of the subspace $(\mathcal{M}_\lambda, \lambda^\star)$. In Example 3.1, there is no $g^\star_\lambda-$closed subset and here also, the two concepts coincide. The following Theorem 3.1 gives some properties of $g^\star_\lambda-$closed sets. Example 3.4 shows that the converse of Theorem 3.1(a) is not true.

Theorem 3.1. *Let (X, μ) be a generalized space and $A \subset X$. Then the following hold.*
(a) If A is a $\lambda-$closed subset of X, then $A \cap \mathcal{M}_\lambda$ is a $g^\star_\lambda-$closed set.
(b) $c_\lambda(A) \cap \mathcal{M}_\lambda$ is a $g^\star_\lambda-$closed set for every subset A of X.

Proof: (a) Let $A\cap\mathcal{M}_\lambda \subset M$ and M be $\lambda-$open. Since $c_\lambda(A\cap\mathcal{M}_\lambda)\cap\mathcal{M}_\lambda = c_\lambda(A)\cap \mathcal{M}_\lambda$, by Lemma 3.1(b), we have $c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda = c_\lambda(A) \cap \mathcal{M}_\lambda = A \cap \mathcal{M}_\lambda \subset M$. Therefore, we have $c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda \subset M$ and so $A \cap \mathcal{M}_\lambda$ is $g^\star_\lambda-$closed.
(b) The proof follows from (a). □

Example 3.4. *Let $X = \{a,\ b,\ c\}$ and $\mu = \{\emptyset,\ \{a\},\ \{b\},\ \{a,\ b\}\}$. Then $\mu-$closed sets are X, $\{a,\ c\}$, $\{b,\ c\}$ and $\{c\}$. If $A = \{a,\ b\}$, then $A \cap \mathcal{M}_\mu = \{a,\ b\}$ and $A \cap \mathcal{M}_\mu$ is a $g^\star_\mu-$closed set but A is not $\mu-$closed.*

The following Theorem 3.2 gives a characterization of $g^\star_\lambda-$closed sets.

Theorem 3.2. *Let (X, μ) be a space. Then a subset A of $\mathcal{M}_\lambda$ is $g^\star_\lambda-$closed if and only if $F \subset c_\lambda(A) - A$ and F is $\lambda-$closed imply that $F = X - \mathcal{M}_\lambda$.*

Proof: Let F be a $\lambda-$closed subset of $c_\lambda(A) - A$. Since $A \subset X - F$ and A is $g^\star_\lambda-$closed, $c_\lambda(A) \cap \mathcal{M}_\lambda \subset X - F$ and so $F \subset X - (c_\lambda(A) \cap \mathcal{M}_\lambda) = (X - c_\lambda(A) \cup (X - \mathcal{M}_\lambda)$. Since $F \subset c_\lambda(A)$, we have $F \subset (X - \mathcal{M}_\lambda)$. Therefore, by Lemma 3.1(a), $F = X - \mathcal{M}_\lambda$. Conversely, suppose the condition holds and $A \subset M$ and $M \in \lambda$. Suppose $(c_\lambda(A) \cap \mathcal{M}_\lambda) \cap (X - M)$ is a nonempty subset. Then $(c_\lambda(A) \cap \mathcal{M}_\lambda) \cap (X - M) \subset c_\lambda(A) \cap (X - M) \subset c_\lambda(A) \cap (X - A) \subset c_\lambda(A) - A$. Thus $c_\lambda(A) \cap (X - M)$ is a nonempty $\lambda-$closed set contained in $c_\lambda(A) - A$. Therefore, $c_\lambda(A) \cap (X - M) = X - \mathcal{M}_\lambda$ which implies that $(c_\lambda(A) \cap \mathcal{M}_\lambda) \cap (X - M) = \emptyset$, a contradiction to the assumption. Therefore, $c_\lambda(A) \cap \mathcal{M}_\lambda \subset M$ which implies that A is a $g^\star_\lambda-$closed set. □

Theorem 3.3. *Let (X, μ) be a generalized space. Then a $g^\star_\lambda-$closed subset A of $\mathcal{M}_\lambda$ is a $\lambda-$closed set, if $c_\lambda(A) - A$ is a $\lambda-$closed set.*

Proof: By Theorem 3.2, $c_\lambda(A) - A = X - \mathcal{M}_\lambda$. Then $c_\lambda(A) = A \cup (X - \mathcal{M}_\lambda)$. By Lemma 3.1(i), A is $\lambda-$closed. □

The following Theorem 3.4 shows that in a $\gamma-$space (X, μ), the union of two $g^\star_\delta-$closed sets (resp. $g^\star_\nu-$closed sets) is again a $g^\star_\delta-$closed set (resp. $g^\star_\nu-$closed sets). Example 3.5 shows that the condition $\gamma-$space on the space cannot be replaced by generalized topology. Example 3.6 below shows that the intersection of two $g^\star_\lambda-$closed sets need not be a $g^\star_\lambda-$closed set in a strong generalized space. Theorem 3.5 shows that, the intersection of a $g^\star_\lambda-$closed set with a $\lambda-$closed is a $g^\star_\lambda-$closed set.

Theorem 3.4. *Let (X,μ) be a $\gamma-$space. Then the following hold.*
(a) If A and B are $g_\delta^\star-$closed subsets of $\mathcal{M}_\delta$, then $A\cup B$ is also a $g_\delta^\star-$closed set.
(b) If A and B are $g_\nu^\star-$closed subsets of $\mathcal{M}_\nu$, then $A\cup B$ is also a $g_\nu^\star-$closed set.

Proof: (a) Suppose A and B are $g_\delta^\star-$closed sets. Let $M\in\delta$ such that $A\cup B\subset M$. Since A and B are $g_\delta^\star-$closed sets, $c_\delta(A)\cap\mathcal{M}_\delta\subset M$ and $c_\delta(B)\cap\mathcal{M}_\delta\subset M$ and so $(c_\delta(A)\cap\mathcal{M}_\delta)\cup(c_\delta(B)\cap\mathcal{M}_\delta)\subset M$ and so $(c_\delta(A)\cup c_\delta(B))\cap\mathcal{M}_\delta\subset M$. By Lemma 1.1(c), it follows that $c_\delta(A\cup B)\cap\mathcal{M}_\delta\subset M$ and so the proof follows.
(b) The proof follows from (a) and Lemma 1.1(c). □

Example 3.5. *Let $X=\{a,b,c\}$ and $\mu=\{\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$. Then μ is a GT but not a quasi-topology. If $A=\{b\}$ and $B=\{c\}$, then A and B are $g_\delta^\star-$closed sets but their union is not a $g_\delta^\star-$closed set.*

Example 3.6. *Consider the space (X,μ) where $X=\{a,b,c,d,e\}$ with $\mu=\{\emptyset,\{a,b\},\{a,c\},\{a,b,c\},X\}$. If $A=\{a,c,d\}$ and $B=\{b,c,e\}$, then A and B are $g_\delta^\star-$closed sets. But $A\cap B=\{c\}$, is not a $g_\delta^\star-$closed set, since $\{c\}\subset\{a,b,c\}$ but $c_\delta(\{c\})\cap\mathcal{M}_\delta=X$.*

Theorem 3.5. *Let (X,μ) be a generalized space. If A is $g_\lambda^\star-$closed subset of $\mathcal{M}_\lambda$ and B is $\lambda-$closed, then $A\cap B$ is a $g_\lambda^\star-$closed set.*

Proof: Suppose $A\cap B\subset M$ where M is $\lambda-$open. Then $A\subset(M\cup(X-B))$. Since A is $g_\lambda^\star-$closed, $c_\lambda(A)\cap\mathcal{M}_\lambda\subset(M\cup(X-B))$ and so $(c_\lambda(A)\cap B\cap\mathcal{M}_\lambda)=(c_\lambda(A)\cap c_\lambda(B))\cap\mathcal{M}_\lambda\subset M$ which implies that $c_\lambda(A\cap B)\cap\mathcal{M}_\lambda\subset M$ and so $A\cap B$ is a $g_\lambda^\star-$closed set. □

A subset A of $\mathcal{M}_\lambda$ in a space (X,μ) is said to be *$g_\lambda^\star-$open* if $\mathcal{M}_\lambda-A$ is $g_\lambda^\star-$closed. The following Theorem 3.6 gives a characterization of $g_\lambda^\star-$open sets. Since the intersection of two $g_\lambda^\star-$closed sets need not be a $g_\lambda^\star-$closed set, the union of two $g_\lambda^\star-$open sets need not be a $g_\lambda^\star-$open set. Theorem 3.7 below gives a characterization of $g_\lambda^\star-$open sets and Theorem 3.8 below gives a property of $g_\lambda^\star-$open sets. Theorem 3.9 below gives a characterization of $g_\lambda^\star-$closed sets in terms of $g_\lambda^\star-$open sets.

Theorem 3.6. *A subset A of $\mathcal{M}_\lambda$ in a space (X,μ) is $g_\lambda^\star-$open if and only if $F\cap\mathcal{M}_\lambda\subset i_\lambda(A)$ whenever F is $\lambda-$closed and $F\cap\mathcal{M}_\lambda\subset A$.*

Proof: Let A be a $g_\lambda^\star-$open subset of $\mathcal{M}_\lambda$ and F be a $\lambda-$closed subset of X such that $F\cap\mathcal{M}_\lambda\subset A$. Then $M_\lambda-A\subset M_\lambda-(F\cap\mathcal{M}_\lambda)=M_\lambda-F$. Since $M_\lambda-F$ is $\lambda-$open and $M_\lambda-A$ is $g_\lambda^\star-$closed, $c_\lambda(M_\lambda-A)\cap\mathcal{M}_\lambda\subset\mathcal{M}_\lambda-F$ and so $F\subset\mathcal{M}_\lambda-(c_\lambda(\mathcal{M}_\lambda-A)\cap\mathcal{M}_\lambda)=\mathcal{M}_\lambda\cap(\mathcal{M}_\lambda-c_\lambda(\mathcal{M}_\lambda-A))=i_\lambda(A)\cap\mathcal{M}_\lambda=i_\lambda(A)$. Conversely, suppose the condition holds. Let A be a subset of $\mathcal{M}_\lambda$ and F is $\lambda-$closed such that $F\cap\mathcal{M}_\lambda\subset A$. By hypothesis, $F\cap\mathcal{M}_\lambda\subset i_\lambda(A)$ which implies that $\mathcal{M}_\lambda-i_\lambda(A)\subset\mathcal{M}_\lambda-(F\cap\mathcal{M}_\lambda)$ and $c_\lambda(\mathcal{M}_\lambda-A)\subset\mathcal{M}_\lambda-F$. Then $c_\lambda(\mathcal{M}_\lambda-A)\cap\mathcal{M}_\lambda\subset(\mathcal{M}_\lambda-F)\cap\mathcal{M}_\lambda=\mathcal{M}_\lambda-F$ which implies that $\mathcal{M}_\lambda-A$ is $g_\lambda^\star-$closed and so A is $g_\lambda^\star-$open. □

Theorem 3.7. *Let (X,μ) be a space. A subset A of $\mathcal{M}_\lambda$ is $g^\star_\lambda-$open if and only if $M=\mathcal{M}_\lambda$ whenever M is $\lambda-$open and $i_\lambda(A)\cup(\mathcal{M}_\lambda-A)\subset M$.*

Proof: Suppose A is $g^\star_\lambda-$open subset of $\mathcal{M}_\lambda$ and M is $\lambda-$open such that $i_\lambda(A)\cup(\mathcal{M}_\lambda-A)\subset M$. Then $\mathcal{M}_\lambda-M\subset(\mathcal{M}_\lambda-i_\lambda(A))\cap A=c_\lambda(\mathcal{M}_\lambda-A)\cap A=c_\lambda(\mathcal{M}_\lambda-A)-(\mathcal{M}_\lambda-A)$ and so $(\mathcal{M}_\lambda-M)\cup(X-\mathcal{M}_\lambda)\subset c_\lambda(\mathcal{M}_\lambda-A)-(\mathcal{M}_\lambda-A)$. By Theorem 3.2, $(\mathcal{M}_\lambda-M)\cup(X-\mathcal{M}_\lambda)=X-\mathcal{M}_\lambda$ and so $\mathcal{M}_\lambda-M=\emptyset$ which implies that $\mathcal{M}_\lambda=M$. Conversely, suppose the condition holds. Let F be a $\lambda-$closed set such that $F\cap\mathcal{M}_\lambda\subset A$. Since $i_\lambda(A)\cup(\mathcal{M}_\lambda-A)\subset i_\lambda(A)\cup(\mathcal{M}_\lambda-F)\cup(\mathcal{M}_\lambda-\mathcal{M}_\lambda)=i_\lambda(A)\cup(\mathcal{M}_\lambda-F)$ and $i_\lambda(A)\cup(\mathcal{M}_\lambda-F)$ is $\lambda-$open, by hypothesis, $\mathcal{M}_\lambda=i_\lambda(A)\cup(\mathcal{M}_\lambda-F)$ and so $F\cap\mathcal{M}_\lambda\subset(i_\lambda(A)\cup(\mathcal{M}_\lambda-F))\cap F=(i_\lambda(A)\cap F)\cup((\mathcal{M}_\lambda-F)\cap F)=i_\lambda(A)\cap F\subset i_\lambda(A)$. By Theorem 3.6, A is $g^\star_\lambda-$open. □

Theorem 3.8. *Let (X,μ) be a space and A and B be subsets of $\mathcal{M}_\lambda$. If $i_\lambda(A)\subset B\subset A$ and A is $g^\star_\lambda-$open, then B is $g^\star_\lambda-$open.*

Proof: The proof follows from Theorem 3.7. □

Theorem 3.9. *Let (X,λ) be a space. Then a subset A of $\mathcal{M}_\lambda$ is $g^\star_\lambda-$closed if and only if $(c_\lambda(A)-A)\cap\mathcal{M}_\lambda$ is $g^\star_\lambda-$open.*

Proof: Suppose $(c_\lambda(A)-A)\cap\mathcal{M}_\lambda$ is $g^\star_\lambda-$open. Let $A\subset M$ and M is $\lambda-$open. Since $c_\lambda(A)\cap(\mathcal{M}_\lambda-M)\subset c_\lambda(A)\cap(\mathcal{M}_\lambda-A)=(c_\lambda(A)-A)\cap\mathcal{M}_\lambda$, $(c_\lambda(A)-A)\cap\mathcal{M}_\lambda$ is $g^\star_\lambda-$open and $c_\lambda(A)\cap(\mathcal{M}_\lambda-M)$ is $\lambda-$closed, by Theorem 3.6, $c_\lambda(A)\cap(\mathcal{M}_\lambda-M)\subset i_\lambda((c_\lambda(A)-A)\cap\mathcal{M}_\lambda)\subset i_\lambda(c_\lambda(A))\cap i_\lambda(\mathcal{M}_\lambda-A)\subset i_\lambda(c_\lambda(A))\cap i_\lambda(X-A)=i_\lambda(c_\lambda(A))\cap(X-c_\lambda(A))=\emptyset$. Therefore, $c_\lambda(A)\cap\mathcal{M}_\lambda\subset M$ which implies that A is $g^\star_\lambda-$closed. Conversely, suppose A is $g^\star_\lambda-$closed and $F\cap\mathcal{M}_\lambda\subset(c_\lambda(A)-A)\cap\mathcal{M}_\lambda$, where F is $\lambda-$closed. Then $F\subset(c_\lambda(A)-A)$ and so by Theorem 3.2, $F=X-\mathcal{M}_\lambda$ and so $\emptyset=(X-\mathcal{M}_\lambda)\cap\mathcal{M}_\lambda=F\cap\mathcal{M}_\lambda\subset(c_\lambda(A)-A)\cap\mathcal{M}_\lambda$ which implies that $F\cap\mathcal{M}_\lambda\subset i_\lambda((c_\lambda(A)-A)\cap\mathcal{M}_\lambda)$. By Theorem 3.6, $c_\lambda(A)-A$ is $g^\star_\lambda-$open. □

4. R_0 and R_1-spaces

In this section, we define and discuss generalized R_0 and R_1 spaces which are not *strong* and establish that all the results established already will follow as a corollary. Generalized R_0 and R_1 spaces are independently defined by Sivagami and Sivaraj [15], Roy [12] and Sarsak [13]. Unless otherwise stated, in this section, (X,μ) is a generalized space which is not *strong* and $\lambda\in\{\mu,\ \delta,\ \alpha,\ \sigma,\ \pi,\ b,\ \beta,\ \nu,\ \xi,\ \eta,\ \varepsilon,\ \psi\}$. The following definitions and Lemma 4.1 are essential to proceed further. For $A\subset\mathcal{M}_\lambda$, we define $\wedge_\lambda(A)=\cap\{U\subset X\mid A\subset U$ and $U\in\lambda\}$ [15]. The following Lemma 4.1 gives the properties of the operator $\wedge_\lambda$, the proof is similar to the corresponding result in [15].

Lemma 4.1. *[15, Theorem 3.1] Let (X,μ) be a generalized space and A, B and C_ι for $\iota\in\Delta$ be subsets of $\mathcal{M}_\lambda$. Then the following hold.*

(a) If $A \subset B$, then $\wedge_\lambda(A) \subset \wedge_\lambda(B)$.
(b) $A \subset \wedge_\lambda(A)$.
(c) $\wedge_\lambda(\wedge_\lambda(A)) = \wedge_\lambda(A)$.
(d) $\wedge_\lambda(\cup\{C_\iota \mid \iota \in \Delta\}) = \cup\{\wedge_\lambda(C_\iota) \mid \iota \in \Delta\}$.
(e) $\wedge_\lambda(\cap\{C_\iota \mid \iota \in \Delta\}) \subset \cap\{\wedge_\lambda(C_\iota) \mid \iota \in \Delta\}$.
(f) If $A \in \lambda$, then $\wedge_\lambda(A) = A$.
(g) $\wedge_\lambda(A) = \{x \in \mathcal{M}_\lambda \mid c_\lambda(\{x\}) \cap A \neq \emptyset\}$.
(h) For every $x,\ y \in \mathcal{M}_\lambda,\ y \in \wedge_\lambda(\{x\})$ if and only if $x \in c_\lambda(\{y\}) \cap \mathcal{M}_\lambda$.
(i) $\wedge_\lambda(\{x\}) \neq \wedge_\lambda(\{y\})$ if and only if $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$ for every $x,\ y \in \mathcal{M}_\lambda$.

A space (X,λ) is said to be a $\lambda - R_0$ *space* [15,12,13] if every $\lambda-$open subset of X contains the $\lambda-$closure of its singletons. (X,λ) is said to be a $\lambda - R_1$ *space* [15,12,13] if for $x, y \in X$ with $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$, there exist disjoint $\lambda-$open sets G and H such that $c_\lambda(\{x\}) \subset G$ and $c_\lambda(\{y\}) \subset H$. The results on generalized R_0 and R_1 spaces are independently established in [15,12,13]. The space in Example 3.1 is neither $\lambda - R_0$ nor $\lambda - R_1$. Example 3.2 is $\lambda - R_0$, since each point is $\lambda-$closed but is not $\lambda - R_1$, since no disjoint $\lambda-$open sets exist. In particular, if a space is *not strong*, then it is neither $\lambda - R_0$ nor $\lambda - R_1$ (Refer Example 3.1). To rectify it, we redefine R_0 and R_1 spaces as follows.

A generalized space (X,λ) is said to be a $\lambda^\star - R_0$ *space* if for every $\lambda-$open subset G of $\mathcal{M}_\lambda$ and $x \in G,\ c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset G$. (X,λ) is said to be a $\lambda^\star - R_1$ *space* if for $x, y \in \mathcal{M}_\lambda$ with $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$, there exist disjoint $\lambda-$open sets G and H such that $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset G$ and $c_\lambda(\{y\}) \cap \mathcal{M}_\lambda \subset H$. Clearly, for strong spaces, $\lambda^\star - R_i$ spaces coincide with $\lambda - R_i$ spaces and every $\lambda^\star - R_1$ space is a $\lambda^\star - R_0$ space but the converse is not true (Refer to Example 3.2). Also, for $i = 1, 2, (X,\lambda)$ is $\lambda - R_i$ implies that (X,λ) is $\lambda^\star - R_i$. The following Example 4.1 shows that the converses are not true and it shows that non strong generalized spaces may be $\lambda^\star - R_0$ and $\lambda^\star - R_1$ spaces. Theorems in this section give characterizations of $\lambda^\star - R_i, i = 1, 2$ generalized spaces which are true for both strong and non strong generalized spaces.

Example 4.1. *Let* $X = \{a, b, c\}$ *and* $\mu = \{\emptyset, \{a\}, \{b\}, \{a,\ b\}\}$. *Since* $c_\mu(\{a\}) = \{a,\ c\}$ *and* $c_\mu(\{b\}) = \{b,\ c\}$, *it is easy to show that* (X,μ) *is neither* $\mu - R_1$ *nor* $\mu - R_0$ *but* (X,μ) *is both* $\mu^\star - R_1$ *and* $\mu^\star - R_0$.

Theorem 4.1. *For a generalized space* (X,μ), *the following are equivalent.*
(a) (X,λ) *is* $\lambda^\star - R_0$.
(b) For each $\lambda-$*closed set* F *and* $x \notin F$, *there exists* $U \in \lambda$ *such that* $F \cap M_\lambda \subset U$ *and* $x \notin U$.
(c) For every $\lambda-$*closed set* F *with* $x \notin F,\ F \cap c_\lambda(\{x\}) = X - M_\lambda$.
(d) For any two distinct points $x, y \in M_\lambda$, *either* $c_\lambda(\{x\}) = c_\lambda(\{y\})$ *or* $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$.

Proof: (a)⇒(b). Let F be a $\lambda-$closed set and $x \notin F$. Then by hypothesis, $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset X - F$ and so $F \subset (X - c_\lambda(\{x\})) \cup (X - \mathcal{M}_\lambda)$. Therefore, $F \cap \mathcal{M}_\lambda \subset (X - c_\lambda(\{x\})) \cap \mathcal{M}_\lambda \subset X - c_\lambda(\{x\})$. If $U = X - c_\lambda(\{x\})$, then $x \notin U$

and $U \in \lambda$ such that $F \cap \mathcal{M}_\lambda \subset U$.
(b)⇒(c). Let F be a $\lambda-$closed set and $x \notin F$. Then by hypothesis, there exists $U \in \lambda$ such that $x \notin U$ and $F \cap \mathcal{M}_\lambda \subset U$. $x \notin U$ implies that $U \cap c_\lambda(\{x\}) = \emptyset$ and so $(F \cap \mathcal{M}_\lambda) \cap c_\lambda(\{x\}) = \emptyset$ which implies that $F \cap c_\lambda(\{x\}) = X - \mathcal{M}_\lambda$.
(c)⇒(d). Let $x,\ y \in \mathcal{M}_\lambda$ such that $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. Then there exists $z \in c_\lambda(\{x\})$ such that $z \notin c_\lambda(\{y\})$. Then there exists $z \in V \in \lambda$ such that $y \notin V$ and $x \in V$. Hence $x \notin c_\lambda(\{y\})$. By hypothesis, $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - \mathcal{M}_\lambda$.
(d)⇒(a). Let G be a $\lambda-$open set such that $x \in G$. If $y \notin G$, then $x \neq y$ and so $x \notin c_\lambda(\{y\})$ which implies that $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. By hypothesis, $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - \mathcal{M}_\lambda$ and so $y \notin c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$. Hence $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset G$ which implies that (X, λ) is a $\lambda^\star - R_0$ space. □

Theorem 4.2. *Let (X, μ) be generalized space. Then, (X, λ) is a $\lambda^\star - R_0$ space if and only if for $x,\ y \in \mathcal{M}_\lambda$, $\wedge_\lambda(\{x\}) \neq \wedge_\lambda(\{y\})$ implies that $\wedge_\lambda(\{x\}) \cap \wedge_\lambda(\{y\}) = \emptyset$.*

Proof: Suppose (X, λ) is a $\lambda^\star - R_0$ space. Let $x,\ y \in \mathcal{M}_\lambda$ such that $\wedge_\lambda(\{x\}) \neq \wedge_\lambda(\{y\})$. By Lemma 4.1(i), $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. By Theorem 4.1, it follows that $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - \mathcal{M}_\lambda$. Let $z \in \wedge_\lambda(\{x\}) \cap \wedge_\lambda(\{y\})$. Then $z \in \wedge_\lambda(\{x\})$ and $z \in \wedge_\lambda(\{y\})$ and so by Lemma 4.1(h), $x \in c_\lambda(\{z\}) \cap \mathcal{M}_\lambda$ and $y \in c_\lambda(\{z\}) \cap \mathcal{M}_\lambda$ which implies that $\{x, y\} \subset c_\lambda(\{z\})$. Therefore, $c_\lambda(\{x\}) \cup c_\lambda(\{y\}) \subset c_\lambda(\{z\}$. Now $x \in c_\lambda(\{z\}) \cap \mathcal{M}_\lambda$ implies that $x \in c_\lambda(\{x\}) \cap c_\lambda(\{z\}) \cap \mathcal{M}_\lambda$ and so $c_\lambda(\{x\}) \cap c_\lambda(\{z\}) \cap \mathcal{M}_\lambda \neq \emptyset$. By Theorem 4.1(d), $c_\lambda(\{x\}) = c_\lambda(\{z\})$. Similarly, $c_\lambda(\{y\}) = c_\lambda(\{z\})$ and so $c_\lambda(\{x\}) = c_\lambda(\{y\})$, a contradiction. Therefore, $\wedge_\lambda(\{x\}) \cap \wedge_\lambda(\{y\}) = \emptyset$. Conversely, suppose the condition holds. Let $x, y \in X$ such that $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. By Lemma 4.1(i), $\wedge_\lambda(\{x\}) \neq \wedge_\lambda(\{y\})$. By hypothesis, $\wedge_\lambda(\{x\}) \cap \wedge_\lambda(\{y\}) = \emptyset$. We prove that $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - \mathcal{M}_\lambda$. Suppose $z \in \mathcal{M}_\lambda$ such that $z \in c_\lambda(\{x\}) \cap c_\lambda(\{y\})$. Then $z \in c_\lambda(\{x\})$ and $z \in c_\lambda(\{y\})$. Now $z \in c_\lambda(\{x\})$ implies that $x \in \wedge_\lambda(\{z\})$ and so $\wedge_\lambda(\{x\}) \cap \wedge_\lambda(\{z\}) \neq \emptyset$. Similarly, we can prove that $\wedge_\lambda(\{y\}) \cap \wedge_\lambda(\{z\}) \neq \emptyset$. So by hypothesis, $c_\lambda(\{x\}) = c_\lambda(\{y\}) = c_\lambda(\{z\})$, a contradiction. Thus $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - \mathcal{M}_\lambda$. By Theorem 4.1, X is a $\lambda^\star - R_0$ space. □

Theorem 4.3. *For a generalized space (X, μ), the following are equivalent.*
(a) (X, λ) is a $\lambda^\star - R_0$ space.
(b) For any nonempty subset A of $\mathcal{M}_\lambda$ and a $\lambda-$open set G such that $A \cap G \neq \emptyset$, there exists a $\lambda-$closed set F such that $A \cap F \neq \emptyset$ and $F \cap \mathcal{M}_\lambda \subset G$.
(c) If $G \neq \emptyset$ is $\lambda-$open, then $G = \cup\{F \cap \mathcal{M}_\lambda \mid F \cap \mathcal{M}_\lambda \subset G$ and F is $\lambda-$closed$\}$.
(d) If F is $\lambda-$closed, then $F = \cap\{G \cup (X - \mathcal{M}_\lambda) \mid F \subset G \cup (X - \mathcal{M}_\lambda)$ and G is $\lambda-$open$\}$.
(e) For every $x \in \mathcal{M}_\lambda$, $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset \wedge_\lambda(\{x\})$.

Proof: (a)⇒(b). Suppose (X, λ) is a $\lambda^\star - R_0$ space. Let A be a nonempty subset of $\mathcal{M}_\lambda$ and G be a $\lambda-$open set such that $A \cap G \neq \emptyset$. If $x \in A \cap G$, then $x \in G$ and so by hypothesis, $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset G$. If $F = c_\lambda(\{x\})$, then F is the required $\lambda-$closed set such that $A \cap F \neq \emptyset$ and $F \cap \mathcal{M}_\lambda \subset G$.
(b)⇒(c). Let G be $\lambda-$open. Clearly, $G \supset \cup\{F \cap \mathcal{M}_\lambda \mid F \cap \mathcal{M}_\lambda \subset G$ and F is

$\lambda-$closed$\}$. If $x \in G$, then $\{x\} \cap G \neq \emptyset$ and so by (b), there is a $\lambda-$closed set F such that $\{x\} \cap F \neq \emptyset$ and $F \cap \mathcal{M}_\lambda \subset G$ which implies that $x \in \cup\{F \cap \mathcal{M}_\lambda \mid F \cap \mathcal{M}_\lambda \subset G$ and F is $\lambda-$closed$\}$. Therefore, $G \subset \cup\{F \cap \mathcal{M}_\lambda \mid F \cap \mathcal{M}_\lambda \subset G$ and F is $\lambda-$closed$\}$. This completes the proof.
(c)$\Rightarrow$(d). Let F be $\lambda-$closed. By (c), $X - F = \cup\{K \cap \mathcal{M}_\lambda \mid F \subset (X - K) \cup (X - \mathcal{M}_\lambda)$ and K is $\lambda-$closed$\}$ and so $F = \cap\{(X - K) \cup (X - \mathcal{M}_\lambda) \mid F \subset (X - K) \cup (X - \mathcal{M}_\lambda)$ and $X - K$ is $\lambda-$open$\}$$=$$\cap\{G \cup (X - \mathcal{M}_\lambda) \mid F \subset G \cup (X - \mathcal{M}_\lambda)$ and G is $\lambda-$open$\}$.
(d)$\Rightarrow$(e). Let $x \in \mathcal{M}_\lambda$. If $y \notin \wedge_\lambda(\{x\})$, then by Lemma 3.1(g), $\{x\} \cap c_\lambda(\{y\}) = \emptyset$. By (d), $c_\lambda(\{y\}) = \cap\{G \cup (X - \mathcal{M}_\lambda) \mid c_\lambda(\{y\}) \subset G \cup (X - \mathcal{M}_\lambda)$ and G is $\lambda-$open$\}$. Therefore, there is a $\lambda-$open G such that $c_\lambda(\{y\}) \subset G \cup (X - \mathcal{M}_\lambda)$ and $x \notin G$ which implies that $y \notin c_\lambda(\{x\})$. Therefore, $c_\lambda(\{x\}) \subset \wedge_\lambda(\{x\})$.
(e)$\Rightarrow$(a). Let G be a $\lambda-$open set such that $x \in G$. If $y \in c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$, then by (e), $y \in \wedge_\lambda(\{x\})$. Since $\wedge_\lambda(\{x\}) \subset \wedge_\lambda(G) = G$, $y \in G$ and it follows that $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset G$. Hence (X, λ) is a $\lambda^\star - R_0$ space. □

Corollary 4.3A. *For a generalized space (X, μ), the following are equivalent.*
(a) (X, λ) is a $\lambda^\star - R_0$ space.
(b) For every $x \in \mathcal{M}_\lambda$, $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda = \wedge_\lambda(\{x\})$.

Proof: (a)$\Rightarrow$(b). Let $x \in \mathcal{M}_\lambda$. By Theorem 4.3, $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset \wedge_\lambda(\{x\})$. To prove the converse, assume that $y \in \wedge_\lambda(\{x\})$. By Lemma 4.1(h), $x \in c_\lambda(\{y\}) \cap \mathcal{M}_\lambda$ and so $c_\lambda(\{x\}) \subset c_\lambda(\{y\})$ which implies that $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) \neq X - \mathcal{M}_\lambda$. By Theorem 4.1, $c_\lambda(\{x\}) = c_\lambda(\{y\})$ and so $y \in c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$. Hence $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda = \wedge_\lambda(\{x\})$.
(b)$\Rightarrow$(a). The proof follows from Theorem 4.3. □

Theorem 4.4. *For a generalized space (X, μ), the following are equivalent.*
(a) (X, λ) is a $\lambda^\star - R_0$ space.
(b) For each $x,\ y \in \mathcal{M}_\lambda,\ x \in c_\lambda(\{y\}) \cap \mathcal{M}_\lambda \Rightarrow y \in c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$.

Proof: (a)$\Rightarrow$(b). Suppose (X, λ) is a $\lambda^\star - R_0$ space. Let $x \in c_\lambda(\{y\}) \cap \mathcal{M}_\lambda$ and G be a $\lambda-$open set containing y. By hypothesis, $y \in c_\lambda(\{y\}) \cap \mathcal{M}_\lambda \subset G$ and so $x \in G$ which implies that every open set containing y contains x. Therefore, $y \in c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$.
(b)$\Rightarrow$(a). Let G be a $\lambda-$open set containing x. If $y \notin G$, then by hypothesis, $x \notin c_\lambda(\{y\}) \cap \mathcal{M}_\lambda$ and so $y \notin c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$. Hence $c_\lambda(\{x\}) \cap \mathcal{M}_\lambda \subset G$ and so (X, λ) is a $\lambda^\star - R_0$ space. □

Theorem 4.5. *For a generalized space (X, μ), the following are equivalent.*
(a) (X, λ) is a $\lambda^\star - R_0$ space.
(b) If F is a $\lambda-$closed set, then $F \cap \mathcal{M}_\lambda = \wedge_\lambda(F \cap \mathcal{M}_\lambda)$.
(c) If F is a $\lambda-$closed set and $x \in F \cap \mathcal{M}_\lambda$, then $\wedge(\{x\}) \subset F \cap \mathcal{M}_\lambda$.
(d) If $x \in \mathcal{M}_\lambda$, then $\wedge(\{x\}) \subset c_\lambda(\{x\}) \cap \mathcal{M}_\lambda$.

Proof: (a)$\Rightarrow$(b). If (X, λ) is $\lambda^\star - R_0$ and F is $\lambda-$closed, by Theorem 4.3, $F = \cap\{G \cup (X - \mathcal{M}_\lambda) \mid F \subset G \cup (X - \mathcal{M}_\lambda)$ and G is $\lambda-$open$\}$ and so $F \cap \mathcal{M}_\lambda =$

$\cap\{G\cap\mathcal{M}_\lambda) \mid F\cap\mathcal{M}_\lambda\subset G$ and G is $\lambda-$open$\}$=$\wedge_\lambda(F-\mathcal{M}_\lambda)$.
(b)⇒(c). Let $z\in\wedge_\lambda(\{x\})$. Then z is in every $\lambda-$open set containing x. Since $x\in F\cap\mathcal{M}_\lambda$, x is in every $\lambda-$open set containing $F\cap\mathcal{M}_\lambda$ and so z is in every $\lambda-$open set containing $F\cap\mathcal{M}_\lambda$. Therefore, $z\in\wedge_\lambda(F\cap\mathcal{M}_\lambda)=F\cap\mathcal{M}_\lambda$ and so $\wedge(\{x\})\subset F\cap\mathcal{M}_\lambda$.
(c)⇒(d). The proof is clear.
(d)⇒(a). Let $x\in c_\lambda(\{y\})\cap\mathcal{M}_\lambda$. By Lemma 4.1(h), $y\in\wedge_\lambda(\{x\})$ and so by hypothesis, $y\in c_\lambda(\{x\})\cap\mathcal{M}_\lambda$. By Theorem 4.4, (X,λ) is a $\lambda^\star-R_0$ space. □

The following Theorem 4.6 gives a characterization of $\lambda^\star-R_1$ space.

Theorem 4.6. *For a generalized space (X,μ), the following are equivalent.*
(a) (X,λ) is a $\lambda^\star-R_1$ space.
(b) For $x,\ y\in\mathcal{M}_\lambda$ such that $\wedge_\lambda(\{x\})\neq\wedge_\lambda(\{y\})$, there exist disjoint $\lambda-$open sets G and H such that $c_\lambda(\{x\})\cap\mathcal{M}_\lambda\subset G$ and $c_\lambda(\{y\})\cap\mathcal{M}_\lambda\subset H$.

Proof. (a)⇒(b). Let $x,\ y\in\mathcal{M}_\lambda$ such that $\wedge_\lambda(\{x\})\neq\wedge_\lambda(\{y\})$. Then, by Lemma 4.1(i), $c_\lambda(\{x\})\neq c_\lambda(\{y\})$. Since (X,λ) is a $\lambda^\star-R_1$ space, there exist disjoint $\lambda-$open sets G and H such that $c_\lambda(\{x\})\cap\mathcal{M}_\lambda\subset G$ and $c_\lambda(\{y\})\cap\mathcal{M}_\lambda\subset H$.
(b)⇒(a). Let $x,\ y\in\mathcal{M}_\lambda$ such that $c_\lambda(\{x\})\neq c_\lambda(\{y\})$. By Lemma 4.1(i), $\wedge_\lambda(\{x\})\neq\wedge_\lambda(\{y\})$. By hypothesis, there exist disjoint $\lambda-$open sets G and H such that $c_\lambda(\{x\})\cap\mathcal{M}_\lambda\subset G$ and $c_\lambda(\{y\})\cap\mathcal{M}_\lambda\subset H$ and so (X,λ) is a $\lambda^\star-R_1$ space.

5. $G_\mu-$regular generalized spaces

In [11], $\mu g-$regular spaces are defined as follows. Let (X,τ) be a topological space and μ be a generalized topology on X. (X,τ) is said to be a *$\mu g-$regular* space, if for each closed set F and a point $x\notin F$, there exist disjoint $\mu-$open sets U and V such that $x\in U,\ F\subset V$. The space (X,τ) of Example 3.1 with the family of all generalized open sets μ, which is not strong, is not $\mu g-$regular and the space (X,τ) of Example 3.2 (resp. Example 3.3) with the family of all generalized open sets μ, which is strong, is also not $\mu g-$regular. Example 2.4(a) of [11] gives an example of a $\mu g-$regular space. A space (X,λ) is said to be a *$\lambda-$regular* space [10], if for each $x\in\mathcal{M}_\lambda$ and $\lambda-$closed set F such that $x\notin F$, there exist disjoint $\lambda-$open sets U and V such that $x\in U,\ F\cap\mathcal{M}_\lambda\subset V$. The space (X,μ) in Example 3.2 is not a $\mu-$regular space. Spaces (X,μ) in Examples 5.1(a) and (b) below are $\mu-$regular spaces. The following Lemma 5.1 is due to Min [10] where (c) follows from (b).

Lemma 5.1. *Let (X,μ) be a generalized space. Then the following hold.*
(a) (X,λ) is $\lambda-$regular if and only if for each $x\in\mathcal{M}_\lambda$ and $\lambda-$open set U containing x, there is a $\lambda-$open set V containing x such that $x\in V\subset c_\lambda(V)\cap\mathcal{M}_\lambda\subset U$ [10, Theorem 3.12].
(b) If (X,μ) is $\mu-$regular, then every $\mu-$open set is a $\delta(\mu)-$open set [10, Theorem 3.13].
(c) If (X,μ) is $\mu-$regular, then $\alpha(\mu)=\nu(\delta),\ \sigma(\mu)=\xi(\delta),\ \pi(\mu)=\eta(\delta),\ b(\mu)=\varepsilon(\delta)$ and $\beta(\mu)=\psi(\delta)$.

Let X be a nonempty set and μ be a generalized topology on X. The space (X,μ) is said to be $g_\mu-$*regular* if for each pair consisting of a point $x \in \mathcal{M}_\lambda$ and a $g_\mu^\star-$closed set F not containing x, there exist disjoint $\mu-$open sets U and V such that $x \in U$ and $F \subset V$. By Theorem 3.4(a), every $g_\mu-$regular space is a $\mu-$regular space and the following Example 5.1(b) shows that the converse is not true. Example 5.1(c) gives an example of a $g_\lambda-$regular space. Theorem 5.1 below gives a characterization of $g_\lambda-$regular spaces.

Example 5.1. *(a) Let $X = \mathbf{R}$, the set of all real numbers and $\mathbf{Z}$ be the set of all integers. Then $\mu = \wp(\mathbf{R}-\mathbf{Z})$ is a GT on X. Clearly, a subset G of X is $\mu-$open if and only if $G \subset \mathbf{R}-\mathbf{Z}$ and a subset F of X is $\mu-$closed if and only if $F \supset \mathbf{Z}$. Note that $X \notin \mu$, $c_\mu(A) = A \cup \mathbf{Z}$ for every subset A of X. Then (X,μ) is $\mu-$regular.*
(b) Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$. The space (X,μ) is $\mu-$regular. If $A = \{a,c\}$, then A is $g_\mu^\star-$closed. Since b and A are not separated by disjoint $\mu-$open sets, (X,μ) is not $g_\mu-$regular.
(c) Consider the space (X,μ) of Example 3.5. Then (X,μ) is a $g_\mu-$regular space. Note that this space is not strong.

Theorem 5.1. *Let (X,μ) be a generalized space. Then the following are equivalent.*
(a) (X,λ) is $g_\lambda-$regular.
(b) For each $g_\lambda^\star-$open set G and $x \in G$, there exists a $\lambda-$open set U such that $x \in U \subset c_\lambda(U) \cap \mathcal{M}_\lambda \subset G$.

Proof: (a)⇒(b) Suppose (X,λ) is $g_\lambda-$regular. Let G be a $g_\lambda^\star-$open set containing x. Then $\mathcal{M}_\lambda - G$ is a $g_\lambda^\star-$closed set such that $x \notin \mathcal{M}_\lambda - G$. By hypothesis, there exists disjoint $\lambda-$open sets U and V such that $x \in U$ and $\mathcal{M}_\lambda - G \subset V$. Since $U \cap V = \emptyset, c_\lambda(U) \cap V = \emptyset$ and so $c_\lambda(U) \cap \mathcal{M}_\lambda \subset (X-V) \cap \mathcal{M}_\lambda = \mathcal{M}_\lambda - V \subset G$. Thus, there exists a $\lambda-$open set U such that $x \in U \subset c_\lambda(U) \cap \mathcal{M}_\lambda \subset G$.
(b)⇒(a). Suppose the condition holds. Let $x \in X$ and F be a $g_\lambda^\star-$closed set such that $x \notin F$. Then $U = \mathcal{M}_\lambda - F$ is a $g_\lambda^\star-$open set such that $x \in U$. By hypothesis, there exits a $\lambda-$open set V such that $x \in V \subset c_\lambda(V) \cap \mathcal{M}_\lambda \subset U$. Since $c_\lambda(V) \cap \mathcal{M}_\lambda \subset U = \mathcal{M}_\lambda - F$, we have $F = \mathcal{M}_\lambda - (\mathcal{M}_\lambda - F) \subset \mathcal{M}_\lambda - (c_\lambda(V) \cap \mathcal{M}_\lambda) = \mathcal{M}_\lambda - c_\lambda(V) = G$. Then V and G are the required $\lambda-$open sets such that $x \in V$ and $F \subset G$. Therefore, (X,λ) is $g_\lambda-$regular. □

The following Theorem 5.2 gives another characterization of $g_\mu-$regular spaces.

Theorem 5.2. *Let (X,μ) be a generalized space. Then the following are equivalent.*
(a) (X,λ) is a $g_\lambda-$regular space.
(b) For each $g_\lambda^\star-$closed set F and $x \notin F$, there exists $\lambda-$open sets U and V such that $x \in U$, $F \subset V$ and $c_\lambda(U) \cap c_\lambda(V) = X - \mathcal{M}_\lambda$.

Proof: (a)⇒(b). Let F be a $g_\lambda^\star-$closed set and $x \notin F$. Then there exists disjoint $\lambda-$open sets U and V such that $x \in U$ and $F \subset V$. Clearly, $(X - \mathcal{M}_\lambda) \subset c_\lambda(U) \cap c_\lambda(V)$. Moreover, $c_\lambda(U) \cap c_\lambda(V) = (c_\lambda(U) \cap c_\lambda(V)) \cap \mathcal{M}_\lambda \cup (X - \mathcal{M}_\lambda)$, by Lemma 3.1(d) and so $c_\lambda(A) \cap c_\lambda(B) \supset ((U \cap V) \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda) = \emptyset \cup (X - \mathcal{M}_\lambda) = X - \mathcal{M}_\lambda$.

Hence $c_\lambda(A) \cap c_\lambda(B) = X - \mathcal{M}_\lambda$.
(b)⇒(a). Enough to prove that if A and B are $\lambda-$open set such that $c_\lambda(A) \cap c_\lambda(B) = X - \mathcal{M}_\lambda$, then $A \cap B = \emptyset$. Now $\emptyset = (X - \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda = (c_\lambda(A) \cap c_\lambda(B)) \cap \mathcal{M}_\lambda \supset (A \cap B) \cap \mathcal{M}_\lambda = A \cap B$ and so $A \cap B = \emptyset$. Therefore, the proof follows. □

The following Lemma 5.2 follows from the definitions. Corollary 5.2A below follows from Theorem 5.2 and Lemma 5.2.

Lemma 5.2. *Let (X, μ) be a generalized space. Then (X, λ) is $\lambda^\star - R_0$ if and only if every point of $\mathcal{M}_\lambda$ is $g^\star_\lambda-$closed.*

Corollary 5.2A. *Let (X, λ) be an $\lambda^\star - R_0$, $g_\lambda-$regular space. Then the following hold.*
(a) For distinct points x and y of $\mathcal{M}_\lambda$, there exist $\lambda-$open sets U and V such that $x \in U$, $y \in V$ and $c_\lambda(U) \cap c_\lambda(V) = X - \mathcal{M}_\lambda$.
(b) For distinct points x and y of $\mathcal{M}_\lambda$, there exist disjoint $\lambda-$open sets U and V such that $x \in U$ and $y \in V$.

Let X be a nonempty set and μ be a generalized topology on X. A point x is said to be in the $\theta-$closure of A [6], denoted by $c_{\theta(\mu)}(A)$, if $A \cap c_\mu(U) \neq \emptyset$ for every $x \in U \in \mu$. The following Theorem 5.3 gives characterizations of $g_\lambda-$regular spaces in terms of the $\theta-$closure operator.

Theorem 5.3. *Let X be a nonempty set, μ be a generalized topology on X. Then the following are equivalent.*
(a) X is a $g_\lambda-$regular space.
(b) $c_{\theta(\lambda)}(A) \cap \mathcal{M}_\lambda = \cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\}$ for every subset A of $\mathcal{M}_\lambda$.
(c) $c_{\theta(\lambda)}(A) \cap \mathcal{M}_\lambda = A$ for every $g^\star_\lambda-$closed set A.

Proof: (a)⇒(b). Clearly, $A \subset \cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\}$. We first prove that $\cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\} \subset c_{\theta(\lambda)}(A)$. Let $x \in \cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\}$. Suppose $x \notin c_{\theta(\lambda)}(A)$. Then there is a $\lambda-$open set U containing x such that $A \cap c_\lambda(U) = \emptyset$ and so $A \cap U = \emptyset$. Since $X - U$ is a $\lambda-$closed set and hence a $g^\star_\lambda-$closed set containing A, $x \in X - U$, a contradiction. Hence $x \in c_{\theta(\lambda)}(A)$ which implies that $\cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\} \subset c_{\theta(\lambda)}(A)$. Conversely, suppose $x \notin \cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\}$. Then, there exists a $g^\star_\lambda-$closed set F such that $A \subset F$ and $x \in X - F$. Then there exists disjoint $\lambda-$open sets U and V such that $x \in U \subset c_\lambda(U) \subset X - V \subset X - F \subset X - A$. Hence $A \cap c_\lambda(U) = \emptyset$ which implies that $x \notin c_{\theta(\lambda)}(A)$. Hence it follows that $A \subset \cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\}$. Hence $\cap\{F \mid A \subset F$ and F is $g^\star_\lambda-$closed$\} = c_{\theta(\lambda)}(A) \cap \mathcal{M}_\lambda$.
(b)⇒(c). The proof is clear.
(c)⇒(a). Let F be a $g^\star_\lambda-$closed set not containing x. Then $x \notin c_{\theta(\lambda)}(F)$. Then there exists a $\lambda-$open set U containing x such that $F \cap c_\lambda(U) = \emptyset$. Then U and $X - c_\lambda(U)$ are the required disjoint $\lambda-$open sets such that $x \in U$ and $F \subset X - c_\lambda(U)$. Therefore, X is a $g_\lambda-$regular space. □

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