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A New Characterization of the Mathieu Groups of degree 11 and 12 by the Number of Sylow Subgroups

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ABSTRACT: Let G be a finite group with trivial center and $n_p(G)$ be the number of Sylow $p-$subgroup of G. In this paper we prove that if G is a centerless group and $n_p(G)$=$n_p(M)$, where M denotes either of the Mathieu groups M_{11} or M_{12} for every prime $p \in \pi(G)$, then $M \leq G \leq Aut(M)$.

Key Words: Finite Group, simple group, Sylow subgroup.

Contents

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of prime divisors of $|G|$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Throughout this paper, we denote by $n_p(G)$ the number of Sylow $p-$ subgroup of G, that is., $n_p(G)$=$|Syl_p(G)|$. In 1992, Bi [2] showed that $L_2(p^k)$ can be characterized only by the order of normalizer of its Sylow subgroups. In other words, if G is a group and $|N_G(P)| = |N_{L_2(p^k)}(Q)|$, where $P \in Syl_r(G)$ and $Q \in Syl_r(L_2(p^k))$ for every prime r, then $G \cong L_2(p^k)$. This type of characterization is done for the following groups:
$L_2(p^k)$ [2], $L_n(q)$ [3], $S_4(q)$ [4], Alternating groups [5], $U_n(q)$ [6], Janko groups [10], Mathieu simple groups [11] and ${}^2D_n(p^k)$ [9]. Let S be one of the above simple groups. If G is a group and $n_p(G) = n_p(S)$ for every prime number p and $|G| = |S|$, then $|N_G(P)| = |N_S(Q)|$, where $P \in Syl_p(G)$ and $Q \in Syl_p(S)$ for every prime p, hence $G \cong S$. In this work the assumption of $|G| = |S|$ is replaced with $Z(G) = 1$. The main theorem of our paper is as follow:

Main Theorem: Let G be a finite group with trivial center such that $n_p(G)$=$n_p(M)$, for every prime $p \in \pi(G)$, where M denotes one of the Mathieu groups M_{11} or M_{12}. Then $M \leq G \leq Aut(M)$.

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We note that a finite group G is not necessity characterizable by the number of its Sylow subgroups. For example $n_p(A_5) = n_p(A_5 \times D_8)$, for every prime p, where D_8 is the Dihedral group of degree 8, but $A_5 \ncong A_5 \times D_8$. In this paper, G is a finite group with trivial center. All further unexplained notations are standard and the reader is refered to [1], for example.

2. Preliminary Results

In this section some preliminary lemmas are mentioned without proof and are used in the proof of the main theorem of this paper.

Lemma 2.1. *[7] Let G be a finite solvable group and $|G| = m.n$, where $m = p_1^{\alpha_1}...p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1}...q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, 2, ..., s\}$:*

1. *$q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.*
2. *The order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 2.2. *[8] If G is a simple K_3- group, then G is isomorphic to one of the following groups: A_5, A_6, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,17)$, $PSL(3,3)$, $PSU(3,3)$ or $PSU(4,2)$.*

Lemma 2.3. *[14] Let G be a finite group and M be a normal subgroup of G. Then both $n_p(M)$ and $n_p(G/M)$ divide $n_p(G)$ and moreover $n_p(M)\ n_p(G/M) \mid n_p(G)$ for every prime p.*

Lemma 2.4. *[12] Let G be a group and H, $K \leq G$, then $[H,K] \leq K$ if and only if $H \leq N_G(K)$.*

Lemma 2.5. *[13] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*
(1) A_7, A_8, A_9, A_{10}.
(2) M_{11}, M_{11}, J_2.
(3) (a) $L_2(r)$, where r is a prime and satisfies $r^2 - 1 = 2^a.3^b.v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, $v > 3$, is a prime;
(b) $L_2(2^m)$,where $2^m - 1 = u$, $2^m + 1 = 3t^b$,with $m \geq 2$, u, $t$are primes, $t > 3$, $b \geq 1$;
(c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$;
(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

3. Proof of the Main Theorem

In this section, at first we prove that the group M_{11} is characterizable by the number of its Sylow subgroup.

Case 1. Characterization of the group M_{11}

Let G be a finite group with trivial center such that $n_2(G) = n_2(M_{11}) = 495$, $n_3(G) = n_3(M_{11}) = 55$, $n_5(G) = n_5(M_{11}) = 396$ and $n_{11}(G) = n_{11}(M_{11}) = 144$. First we prove that G is an unsolvable group. If G is a solvable group, since $n_{11}(G) = 144$, then $9 \equiv 1 \pmod{11}$ by Lemma 2.1, which is a contradiction. Hence G is an unsolvable group. Since G is a finite group, then it has a chief series. Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq ... \triangleleft N_{r-1} \trianglelefteq N_r = G$, be a chief series of G. Since G is an unsolvable group, then there exists a non-negative integer i, such that N_i/N_{i-1} is the direct product of isomorphic simple groups and N_{i-1} is a maximal solvable normal subgroup of G. Now set $N_i := H$ and $N_{i-1} := N$. Hence G has the following normal series:

$$1 \trianglelefteq N \triangleleft H \trianglelefteq G$$

such that H/N is a direct product of isomorphic non-abelian simple groups. Let $H/N = S_1 \times \ldots \times S_r$, where S_1 is an unsolvable simple group and $S_1 \cong \ldots \cong S_r$, S_1 is a simple K_3−group or a simple K_4−group. Since $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$ by Lemma 2.3, then H/N is not simple K_3−group, by Lemma 2.2. So H/N is simple K_4−group. If $H/N \cong L_2(r)$, where r is a prime and satisfies $r^2 - 1 = 2^a.3^b.v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$ and $v > 3$, is a prime, then by $\pi(H/N) = \{2, 3, 5, 11\}$, $r = 11$, which is a contradiction by Lemma 2.3. If $H/N \cong L_2(2^m)$ where $2^m - 1 = u$, $2^m + 1 = 3t^b$, with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$, then u, $t \in \{3, 5, 11\}$, which is a contradiction. If $H/N \cong L_2(3^m)$ where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$, then u, $t \in \{3, 5, 11\}$, which is a contradiction. Hence by the above discussion, Lemma 2.5 and [1] $H/N \cong M_{11}$. Now set $\overline{H} := H/N \cong M_{11}$and $\overline{G} := G/N$. On the other hand, we have:

$$M_{11} \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq Aut(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $M_{11} \leq G/K \leq Aut(M_{11})$. Since $Out(M_{11}) = 1$, we have G/K is isomorphic to M_{11}. We prove that $K = N$. Suppose that $K \neq N$, by Lemma 2.3 we have $n_p(K) = 1$ for every prime $p \in \pi(G)$, then K is a nilpotent subgroup of G. On the other hand, since $C_{\overline{G}}(\overline{H}) \cong K/N$ and N is a maximal solvable normal subgroup of G, we have K is an unsolvable normal subgroup of G, which is a contradiction. Thus $K = N$, and so $G/N \cong M_{11}$. We claim that $N = 1$. Let Q be a Sylow $q-$ subgroup of N, since N is nilpotent, then Q is normal in G. Now if $P \in Syl_p(G)$, then Q normalizes P and so if $p \neq q$, then $P \leq N_G(Q) = G$. Also we note that NP/N is a Sylow p-subgroup of G/N. On the other hand, if $R/N = N_{G/N}(NP/N)$, then $R = N_G(P)N$. We know that $n_p(G) = n_p(G/N)$, so $|G : R| = |G : N_G(P)|$. Thus $R = N_G(P)$ and therefore $N \leq N_G(P)$. So $Q \leq N_G(P)$. Since $P \leq N_G(Q)$

and $Q \leq N_G(P)$ by Lemma 2.4, this implies that $[P,Q] \leq P$ and $[P,Q] \leq Q$, so $[P,Q] \leq P \cap Q = 1$. Therefore, $P \leq C_G(Q)$ and $Q \leq C_G(P)$, in the other word P and Q centralize each other. Let $C = C_G(Q)$, then C contains a full Sylow $p-$subgroup of G for all primes p different from q, and thus $|G : C|$ is a power of q. Now let S be a Sylow $q-$subgroup of G. Then $G = CS$. Also if $Q > 1$, then $C_Q(S)$ is nontrivial, and we see that $C_Q(S) \leq Z(G)$. By the assumption $Z(G) = 1$, it follows that $Q = 1$. Since q was arbitary, we have $N = 1$. Therefore, $G \cong M_{11}$.

Case 2. Characterization of the group M_{12}
Let G be a finite group with trivial center such that $n_2(G) = n_2(M_{12}) = 1485$, $n_3(G) = n_3(M_{12}) = 880$, $n_5(G) = n_5(M_{12}) = 2376$ and $n_{11}(G) = n_{11}(M_{12}) = 1728$. At first we prove that G is an unsolvable group. If G is a solvable group, since $n_3(G) = 880$, then $11 \equiv 1 \pmod 3$ by Lemma 2.1, which is a contradiction. Hence G is an unsolvable group. Since G is a finite group and unsolvable, then it has the following normal series:

$$1 \trianglelefteq N \triangleleft H \trianglelefteq G$$

such that H/N is a direct product of isomorphic non-abelian simple groups. Let $H/N = S_1 \times \ldots \times S_r$, where S_1 is an unsolvable simple group and $S_1 \cong \ldots \cong S_r$, S_1 is a simple K_3-group or a simple K_4-group. Let S_1 be a simple K_3-group, since $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$ by Lemma 2.3, then H/N is isomorphic to A_5 or A_6 by Lemma 2.2. Suppose that $H/N \cong A_5$. Similar to the proof of Case 1, there exists a normal subgroup K such that $N \leq K$ and $A_5 \leq G/K \leq S_5$. If $G/K \cong A_5$, then we prove that $K = N$. Suppose that $K \neq N$. Since $N < K$ and N is a maximal solvable normal subgroup G, then K is an unsolvable normal subgroup of G. Hence K has the following normal series:

$$1 \trianglelefteq N_1 \triangleleft H_1 \trianglelefteq K,$$

such that H_1/N_1 is a simple K_3-group or a simple K_4-group. On the other hand, $n_3(K) \mid 88$, $n_2(K) \mid 297$, $n_5(K) \mid 396$ and $n_{11}(K) \mid 1728$ by Lemma 2.3. Since $n_p(H_1/N_1) \mid n_p(K)$ for every prime $p \in \pi(G)$, we get a contradiction by Lemma 2.2 and Lemma 2.5. Therefore, $N = K$ and $G/N \cong A_5$. This implies that $11 \in \pi(N)$ and the order of a Sylow 11-subgroup in G and N are equal. As N is normal in G thus the number of Sylow 11-subgroups of G and N are equal. Therefore the number of Sylow 11-subgroups of N is $1728 = 2^6 \times 3^3$. Since N is solvable by Lemma 2.1, $3^3 \equiv 1 \pmod{11}$, a contradiction. Similar to the above discussion if $G/K \cong S_5$, we can get a contradiction. If $G/K \cong A_6$, then similarly we can get a contradiction. Now let S_1 be a simple K_4-group. It is easy to see by Lemma 2.3, Lemma 2.5 and [1], to conclude that $H/N \cong M_{12}$. Similar to the proof of Case 1, there exists a normal subgroup K such that $N \leq K$ and $M_{12} \leq G/K \leq Aut(M_{12})$. Since $Out(M_{12}) = 2$, we have G/K is isomorphic to one of the groups M_{12} or $M_{12}.2$. Let $G/K \cong M_{12}$, then we prove that $K = N$. Suppose that $K \neq N$, by Lemma 2.3, we have $n_p(K) = 1$ for every prime $p \in \pi(G)$, then K is a nilpotent subgroup of G. On the other hand, since $C_{\overline{G}}(\overline{H}) \cong K/N$ and N

is a maximal solvable normal subgroup of G, we have K is an unsolvable normal subgroup of G, which is a contradiction. Thus $K = N$, and so $G/N \cong M_{12}$. We claim that $N = 1$. Let Q be a Sylow $q-$subgroup of N, since N is nilpotent, then Q is normal in G. We have that if $P \in Syl_p(G)$, then Q normalizes P and so if $p \neq q$, then P and Q centralize each other. Let $C = C_G(Q)$, then C contains a full Sylow $p-$subgroup of G for all primes p different from q, and thus $|G : C|$ is a power of q. Now let S be a Sylow $q-$subgroup of G. Then $G = CS$. Also if $Q > 1$, then $C_Q(S)$ is nontrivial, and we see that $C_Q(S) \leq Z(G)$. By the assumption $Z(G) = 1$, it follows that $Q = 1$. Since q was arbitary, we have $N = 1$. Therefore, $G \cong M_{12}$. If $G/K \cong M_{12}.2$, similarly we can prove that $G \cong M_{12}.2$ and the proof is completed.

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References

1. J. H. Conway, R. T. Curtis, S. P. Norton, et al, *Atlas of finite groups*, Clarendon, Oxford, (1985).
2. J. Bi, *A characterization of $L_2(q)$*, (Chinese), J. Liaoning Univ (Natural Sciences Edition), 19 (2), 1-4 (1992).
3. J. Bi, *A characterization of $L_n(q)$ by the normalizers' orders of their Sylow subgroups*, Acta Math. Sinica (New Ser), 11 (3), 300-306 (1995).
4. J. Bi, *On the group with the same orders of Sylow normalizers as the finite simple group $S_4(q)$*, Algebra, Groups and Geom, 18 (3), 349-355 (2001).
5. J. Bi, *Characterization of Alternating groups by orders of normalizers of Sylow subgroups*, Algebra Colloq, 8 (3), 249-256 (2001).
6. J. Bi, *On the groups with the same orders of Sylow normalizers as the finite projective special unitary group*, Sci. China, Ser A, 47 (6), 801-811 (2004).
7. M. Hall, *The Theory of Groups*, Macmillan, New York, (1959).
8. M. Herzog, *On finite simple groups of order divisible by three primes only*, Journal of Algebra, **120** (10), 383-388 (1968).
9. A. Iranmanesh and N. Ahanjideh, *A characterization of $^2D_n(p^k)$ by order of normalizer of Sylow subgroups*, International journal of Algebra, Vol. 2, no. 18, 853-865 (2008).
10. B. Khosravi and B. Khosravi, *Groups with the same orders of Sylow normalizers as the Janko groups*, J. Apple. Algebra Discrete Struct, 3, 23-31 (2005).
11. B. Khosravi and B. Khosravi, *Two new characterizations of the Mathieu simple groups*, Internal. J. Math. and Mathematical Science, 9, 1449-1453 (2005).
12. D. J. S. Robinson, *A course on the theory of groups*, Springer-Verlag, New York, (1982).
13. W. Shi, *On simple K_4-groups*, Chinese Science Bull, 36(17): 1281–1283 (1991) (in Chinese).
14. J. Zhang, *Sylow numbers of finite groups*, Journal of Algebra, **176**(10), 111-123 (1995).

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