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# Biharmonic $\mathcal{S}$-Curves According to Sabban Frame in Heisenberg Group $\mathrm{Heis}^3$

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ABSTRACT: In this paper, we study biharmonic curves accordig to Sabban frame in the Heisenberg group $\mathrm{Heis}^3$. We characterize the biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group $\mathrm{Heis}^3$. Finally, we find out their explicit parametric equations according to Sabban Frame.

Key Words: Biharmonic curve, Heisenberg group, curvature, torsion.

## Contents



## 1. Introduction

Harmonic maps $f:(M,g)\longrightarrow(N,h)$ between manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M e(f)\,v_g, \tag{1.1}$$

where $v_g$ is the volume form on $(M,g)$ and

$$e(f)(x):=\frac{1}{2}\|df(x)\|^2_{T^*M\otimes f^{-1}TN}$$

is the energy density of f at the point $x\in M$.

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map f is harmonic if and only if its tension field $\tau(f)$ vanishes identically, where the tension field is given by

$$\tau(f)=\mathrm{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [6], we can define the bienergy of a map $f$ by

$$E_2(f)=\frac{1}{2}\int_M \|\tau(f)\|^2\,v_g, \tag{1.3}$$

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and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [7,8], showing that the Euler–Lagrange equation associated to $E_2$ is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of $f$ . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

This study is organised as follows: Firstly, we study biharmonic curves accordig to Sabban frame in the Heisenberg group $\mathrm{Heis}^3$. Secondly, we characterize the biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group $\mathrm{Heis}^3$. Finally, we find out their explicit parametric equations according to Sabban Frame.

## 2. The Heisenberg Group $\mathrm{Heis}^3$

Heisenberg group $\mathrm{Heis}^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \frac{1}{2}\overline{x}y + \frac{1}{2}x\overline{y}) \tag{2.1}$$

$\mathrm{Heis}^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric $g$ is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of $\mathrm{Heis}^3$ has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

We obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 &= -\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1. \end{aligned}$$

The components $\{R_{ijkl}\}$ of $R$ relative to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are defined by

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l) = g\left(R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_l, \mathbf{e}_k\right).$$

The non vanishing components of the above tensor fields are

$$R_{121} = \tfrac{3}{4}\mathbf{e}_2, \quad R_{131} = -\tfrac{1}{4}\mathbf{e}_3, \quad R_{122} = -\tfrac{3}{4}\mathbf{e}_1,$$

$$R_{232} = -\tfrac{1}{4}\mathbf{e}_3, \quad R_{133} = \tfrac{1}{4}\mathbf{e}_1, \quad R_{233} = \tfrac{1}{4}\mathbf{e}_2,$$

and

$$R_{1212} = -\tfrac{3}{4}, \quad R_{1313} = R_{2323} = \tfrac{1}{4}. \tag{2.3}$$

## 3. Biharmonic $\mathcal{S}$-Curves According To Sabban Frame In The Heisenberg Group Heis$^3$

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis$^3$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis$^3$ along $\gamma$ defined as follows:

$\mathbf{T}$ is the unit vector field $\gamma'$ tangent to $\gamma$, $N$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to $\gamma$), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion,

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \; g(\mathbf{N}, \mathbf{N}) = 1, \; g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \longrightarrow \mathbb{S}^2_{Heis^3}$ be unit speed spherical curve. We denote $\sigma$ as the arc-length parameter of $\alpha$. Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of $\alpha$. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along $\alpha$. This frame is called the Sabban frame of $\alpha$ on the Heisenberg group Heis$^3$. Then we have the following spherical Frenet-Serret formulae of $\alpha$ :

$$\begin{aligned} \nabla_{\mathbf{t}}\alpha &= \mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{t} &= -\alpha + \kappa_g\mathbf{s}, \\ \nabla_{\mathbf{t}}\mathbf{s} &= -\kappa_g\mathbf{t}, \end{aligned} \tag{3.2}$$

where $\kappa_g$ is the geodesic curvature of the curve $\alpha$ on the $\mathbb{S}^2_{Heis^3}$ and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\alpha,\alpha)=1,\ g(\mathbf{s},\mathbf{s})=1, \\ g(\mathbf{t},\alpha) &= g(\mathbf{t},\mathbf{s})=g(\alpha,\mathbf{s})=0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned} \alpha &= \alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+\alpha_3\mathbf{e}_3, \\ \mathbf{t} &= t_1\mathbf{e}_1+t_2\mathbf{e}_2+t_3\mathbf{e}_3, \\ \mathbf{s} &= s_1\mathbf{e}_1+s_2\mathbf{e}_2+s_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

To separate a biharmonic curve according to Sabban frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathcal{S}$-curve.

**Theorem 3.1.** *$\alpha : I \longrightarrow \mathbb{S}^2_{Heis^3}$ is a biharmonic $\mathcal{S}$-curve if and only if*

$$\begin{aligned} \kappa_g &= \textit{constant} \neq 0, \\ 1+\kappa_g^2 &= -[\frac{1}{4}-s_3^2]+\kappa_g[-\alpha_3 s_3], \\ \kappa_g''-\kappa_g^3 &= \alpha_3 s_3+\kappa_g[\frac{1}{4}-\alpha_3^2]. \end{aligned} \tag{3.4}$$

**Proof:** Using (2.1) and Sabban formulas (3.2), we have (3.4). □

**Corollary 3.2.** *$\alpha : I \longrightarrow \mathbb{S}^2_{Heis^3}$ is a biharmonic $\mathcal{S}$-curve if and only if*

$$\begin{aligned} \kappa_g &= \textit{constant} \neq 0, \\ 1+\kappa_g^2 &= -[\frac{1}{4}-s_3^2]+\kappa_g[-\alpha_3 s_3], \\ \kappa_g^3 &= -\alpha_3 s_3-\kappa_g[\frac{1}{4}-\alpha_3^2]. \end{aligned} \tag{3.4}$$

**Lemma 3.3.** *All of biharmonic $\mathcal{S}$-curves in $\mathbb{S}^2_{Heis^3}$ are helices.*

**Theorem 3.4.** *Let $\alpha : I \longrightarrow \mathbb{S}^2_{Heis^3}$ be a unit speed non-geodesic biharmonic*

*$\mathcal{S}$-curve. Then, the parametric equations of $\alpha$ are*

$$\begin{aligned}
x^{\mathcal{S}}(\sigma) &= -\frac{\sin^2\varphi}{(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)}\cos[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]+\mathcal{M}_2,\\
y^{\mathcal{S}}(\sigma) &= \frac{\sin^2\varphi}{(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)}\sin[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]+\mathcal{M}_3, \qquad (3.5)\\
z^{\mathcal{S}}(\sigma) &= \cos\varphi\sigma-\sin\varphi\frac{(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)\sigma+\mathcal{M}_1}{2(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)}\\
&\quad -\sin^2\varphi\frac{\sin 2[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]}{4(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)}\\
&\quad +\frac{\mathcal{M}_2}{(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)}\sin^3\varphi\sin[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]+\mathcal{M}_4,
\end{aligned}$$

*where $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\mathcal{M}_4$ are constants of integration.*

**Proof:** Since $\alpha$ is biharmonic, $\alpha$ is a $\mathcal{S}-$helix. So, without loss of generality, we take the axis of $\alpha$ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{t},\mathbf{e}_3)=t_3=\cos\varphi, \qquad (3.6)$$

where $\varphi$ is constant angle.

So, substituting the components $t_1$, $t_2$ and $t_3$ in the equation (3.3), we have the following equation

$$\mathbf{t}=\sin\varphi\sin\mu\mathbf{e}_1+\sin\varphi\cos\mu\mathbf{e}_2+\cos\varphi\mathbf{e}_3. \qquad (3.7)$$

The covariant derivative of the vector field $t$ is:

$$\nabla_{\mathbf{t}}\mathbf{t}=(t_1'+t_2t_3)\mathbf{e}_1+(t_2'-t_1t_3)\mathbf{e}_2+t_3'\mathbf{e}_3. \qquad (3.8)$$

Frrom above equation we have

$$\mu(\sigma)=(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1, \qquad (3.10)$$

where $\mathcal{M}_1$ is a constant of integration.

Thus (3.9) and (3.10), imply

$$\begin{aligned}
\mathbf{t} &= \sin\varphi\sin[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]\mathbf{e}_1 \qquad (3.11)\\
&\quad +\sin\varphi\cos[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]\mathbf{e}_2+\cos\varphi\mathbf{e}_3.
\end{aligned}$$

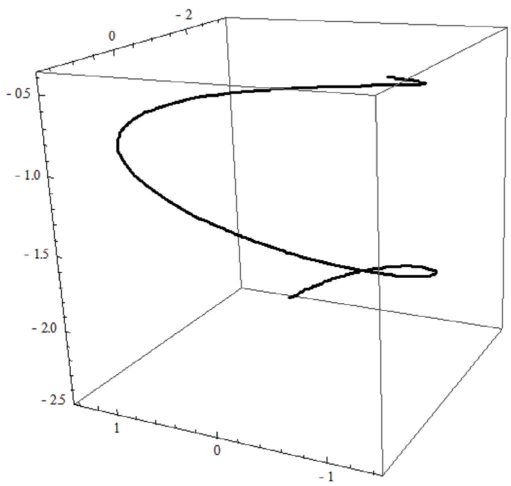
Using (2.1) in (3.11), we obtain

$$\begin{aligned}\mathbf{t} &= (\sin\varphi\sin[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1],\sin\varphi\cos[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1],\\ &\quad \cos\varphi+\sin\varphi(-\frac{\sin^2\varphi}{(\sqrt{1+\kappa_g^2}-\sin\varphi\cos\varphi)}\cos[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1] \qquad (3.12)\\ &\quad +\mathcal{M}_2)\cos[(\frac{\sqrt{1+\kappa_g^2}}{\sin\varphi}-\cos\varphi)\sigma+\mathcal{M}_1]).\end{aligned}$$

where $\mathcal{M}_1,\mathcal{M}_2$ are constants of integration.

Integrating both sides, we have (3.9). This proves our assertion. Thus, the proof of theorem is completed. □

We can use Mathematica in above theorem, yields

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## References

1. M. Babaarslan and Y. Yayli: *The characterizations of constant slope surfaces and Bertrand curves,* International Journal of the Physical Sciences 6(8) (2011), 1868-1875.

2. R. Caddeo and S. Montaldo: *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12(8) (2001), 867–876.

3. B. Y. Chen: *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math. 17 (1991), 169–188.

4. I. Dimitric: *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.

5. J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1–68.

6. J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

7. G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

8. G. Y. Jiang: *2-harmonic maps and their first and second variational formulas,* Chinese Ann. Math. Ser. A 7(4) (1986), 389–402.

9. T. Körpınar and E. Turhan: *On Spacelike Biharmonic Slant Helices According to Bishop Frame in the Lorentzian Group of Rigid Motions* $\mathbb{E}(1,1)$*,* Bol. Soc. Paran. Mat. 30 (2) (2012), 91–100.

10. E. Loubeau and S. Montaldo: *Biminimal immersions in space forms,* preprint, 2004, math.DG/0405320 v1.

11. B. O'Neill: *Semi-Riemannian Geometry*, Academic Press, New York (1983).

12. K. Onda: *Lorentz Ricci Solitons on 3-dimensional Lie groups*, Geom Dedicata 147 (1) (2010), 313-322.

13. E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space* $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 99–104.

14. E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis*$^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

15. E. Turhan and T. Körpınar: *On Characterization Canal Surfaces around Timelike Horizontal Biharmonic Curves in Lorentzian Heisenberg Group Heis*$^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 66a (2011), 441-449.

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