Biharmonic $S$-Curves According to Sabban Frame in Heisenberg Group $\textnormal{Heis}^3$

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ABSTRACT: In this paper, we study biharmonic curves according to Sabban frame in the Heisenberg group $\textnormal{Heis}^3$. We characterize the biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group $\textnormal{Heis}^3$. Finally, we find out their explicit parametric equations according to Sabban Frame.

Key Words: Biharmonic curve, Heisenberg group, curvature, torsion.

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1. Introduction

Harmonic maps $f : (M, g) \rightarrow (N, h)$ between manifolds are the critical points of the energy

$$E(f) = \frac{1}{2} \int_M e(f) \, v_g,$$  \hspace{1cm} (1.1)

where $v_g$ is the volume form on $(M, g)$ and

$$e(f)(x) := \frac{1}{2} \|df(x)\|^2_{T^* M \otimes f^{-1} T N}$$

is the energy density of $f$ at the point $x \in M$.

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map $f$ is harmonic if and only if its tension field $\tau(f)$ vanishes identically, where the tension field is given by

$$\tau(f) = \text{trace} \nabla df.$$  \hspace{1cm} (1.2)

As suggested by Eells and Sampson in [6], we can define the bienergy of a map $f$ by

$$E_2(f) = \frac{1}{2} \int_M \|\tau(f)\|^2 \, v_g,$$  \hspace{1cm} (1.3)

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and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in \[7,8\], showing that the Euler–Lagrange equation associated to \(E_2\) is
\[
\tau_2 (f) = -\mathcal{J}^f (\tau (f)) = -\Delta \tau (f) - \text{trace} R^N (df, \tau (f)) df = 0,
\]
(1.4)
where \(\mathcal{J}^f\) is the Jacobi operator of \(f\). The equation \(\tau_2 (f) = 0\) is called the biharmonic equation. Since \(\mathcal{J}^f\) is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

This study is organised as follows: Firstly, we study biharmonic curves according to Sabban frame in the Heisenberg group \(\text{Heis}^3\). Secondly, we characterize the biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group \(\text{Heis}^3\). Finally, we find out their explicit parametric equations according to Sabban Frame.

2. The Heisenberg Group \(\text{Heis}^3\)

Heisenberg group \(\text{Heis}^3\) can be seen as the space \(\mathbb{R}^3\) endowed with the following multiplication:
\[
(x, y, z)(x, y, z) = (x + x, y + y, z + z - \frac{1}{2}xy + \frac{1}{2}y) \quad (2.1)
\]
\(\text{Heis}^3\) is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric \(g\) is given by
\[
g = dx^2 + dy^2 + (dz - xdy)^2.
\]
The Lie algebra of \(\text{Heis}^3\) has an orthonormal basis
\[
e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \quad (2.2)
\]
for which we have the Lie products
\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0
\]
with
\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]
We obtain
\[
\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0,
\]
\[
\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3,
\]
\[
\nabla_{e_1} e_3 = \nabla_{e_2} e_1 = -\frac{1}{2} e_2,
\]
\[
\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1.
\]
The components \( \{ R_{ijkl} \} \) of \( R \) relative to \( \{ e_1, e_2, e_3 \} \) are defined by
\[
R_{ijk} = R(e_i, e_j) e_k, \quad R_{ijkl} = R(e_i, e_j, e_k, e_l) = g(R(e_i, e_j)e_k, e_l).
\]
The non-vanishing components of the above tensor fields are
\[
R_{121} = \frac{3}{4} e_2, \quad R_{131} = -\frac{1}{4} e_3, \quad R_{122} = -\frac{3}{4} e_1,
\]
\[
R_{232} = -\frac{1}{4} e_3, \quad R_{133} = \frac{1}{4} e_1, \quad R_{233} = \frac{1}{4} e_2,
\]
and
\[
R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}.
\]

3. Biharmonic S-Curves According To Sabban Frame In The Heisenberg Group \( \text{Heis}^3 \)

Let \( \gamma : I \rightarrow \text{Heis}^3 \) be a non geodesic curve on the Heisenberg group \( \text{Heis}^3 \) parametrized by arc length. Let \( \{ T, N, B \} \) be the Frenet frame fields tangent to the Heisenberg group \( \text{Heis}^3 \) along \( \gamma \) defined as follows:
\( T \) is the unit vector field \( \gamma' \) tangent to \( \gamma \), \( N \) is the unit vector field in the direction of \( \nabla_T T \) (normal to \( \gamma \)), and \( B \) is chosen so that \( \{ T, N, B \} \) is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:
\[
\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B, \quad \nabla_T B = -\tau N,
\]
where \( \kappa \) is the curvature of \( \gamma \) and \( \tau \) is its torsion,
\[
g(T, T) = 1, \quad g(N, N) = 1, \quad g(B, B) = 1,
\]
\[
g(T, N) = g(T, B) = g(N, B) = 0.
\]

Now we give a new frame different from Frenet frame. Let \( \alpha : I \rightarrow S^2_{\text{Heis}^3} \) be unit speed spherical curve. We denote \( \sigma \) as the arc-length parameter of \( \alpha \). Let us denote \( t(\sigma) = \alpha'(\sigma) \), and we call \( t(\sigma) \) a unit tangent vector of \( \alpha \). We now set a vector \( s(\sigma) = \alpha(\sigma) \times t(\sigma) \) along \( \alpha \). This frame is called the Sabban frame of \( \alpha \) on the Heisenberg group \( \text{Heis}^3 \). Then we have the following spherical Frenet-Serret formulae of \( \alpha \):
\[
\nabla_t \alpha = t, \quad \nabla_t t = -\alpha + \kappa_s s, \quad \nabla_t s = -\kappa_t t.
\]
where $\kappa_g$ is the geodesic curvature of the curve $\alpha$ on the $\mathcal{S}^2_{\text{Heis}^3}$ and
\[
g(t,t) = 1, \quad g(\alpha,\alpha) = 1, \quad g(s,s) = 1, \\
g(t,\alpha) = g(t,s) = g(\alpha,s) = 0.
\]

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write
\[
\begin{align*}
\alpha &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \\
t &= t_1 e_1 + t_2 e_2 + t_3 e_3, \\
s &= s_1 e_1 + s_2 e_2 + s_3 e_3.
\end{align*}
\tag{3.3}
\]

To separate a biharmonic curve according to Sabban frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $S$-curve.

**Theorem 3.1.** $\alpha : I \rightarrow \mathcal{S}^2_{\text{Heis}^3}$ is a biharmonic $S$-curve if and only if
\[
\begin{align*}
\kappa_g &= \text{constant} \neq 0, \\
1 + \kappa_g^2 &= -\left[\frac{1}{4} - \alpha_3^2\right] + \kappa_g[-\alpha_3 s_3], \\
\kappa_g'' - \kappa_g^3 &= \alpha_3 s_3 + \kappa_g[\frac{1}{4} - \alpha_3^2].
\end{align*}
\tag{3.4}
\]

**Proof:** Using (2.1) and Sabban formulas (3.2), we have (3.4). \qed

**Corollary 3.2.** $\alpha : I \rightarrow \mathcal{S}^2_{\text{Heis}^3}$ is a biharmonic $S$-curve if and only if
\[
\begin{align*}
\kappa_g &= \text{constant} \neq 0, \\
1 + \kappa_g^2 &= -\left[\frac{1}{4} - s_3^2\right] + \kappa_g[-\alpha_3 s_3], \\
\kappa_g^3 &= -\alpha_3 s_3 - \kappa_g[\frac{1}{4} - \alpha_3^2].
\end{align*}
\tag{3.4}
\]

**Lemma 3.3.** All of biharmonic $S$-curves in $\mathcal{S}^2_{\text{Heis}^3}$ are helices.

**Theorem 3.4.** Let $\alpha : I \rightarrow \mathcal{S}^2_{\text{Heis}^3}$ be a unit speed non-geodesic biharmonic
Then, the parametric equations of $\alpha$ are

\[
x^S(\sigma) = -\frac{\sin^2 \varphi}{(\sqrt{1 + \kappa^2} \sin \varphi)} \cos[\frac{\sqrt{1 + \kappa^2} \sin \varphi}{\sin \varphi} - \cos \varphi \sigma + M_1] + M_2,
\]

\[
y^S(\sigma) = \frac{\sin^2 \varphi}{(\sqrt{1 + \kappa^2} - \sin \varphi \cos \varphi)} \sin[\frac{\sqrt{1 + \kappa^2} \sin \varphi}{\sin \varphi} - \cos \varphi \sigma + M_1] + M_3, \quad (3.5)
\]

\[
z^S(\sigma) = \cos \varphi \sigma - \sin \varphi \left(\frac{\sqrt{1 + \kappa^2} \sin \varphi - \cos \varphi}{2(\sqrt{1 + \kappa^2} - \sin \varphi \cos \varphi)}\right)
- \frac{\sin^2 \varphi}{(\sqrt{1 + \kappa^2} - \sin \varphi \cos \varphi)} \sin[\frac{\sqrt{1 + \kappa^2} \sin \varphi}{\sin \varphi} - \cos \varphi \sigma + M_1] + M_4,
\]

where $M_1, M_2, M_3, M_4$ are constants of integration.

**Proof:** Since $\alpha$ is biharmonic, $\alpha$ is a $S-$curve. So, without loss of generality, we take the axis of $\alpha$ is parallel to the vector $e_3$. Then,

\[
g(t, e_3) = t_3 = \cos \varphi, \quad (3.6)
\]

where $\varphi$ is constant angle.

So, substituting the components $t_1, t_2$ and $t_3$ in the equation (3.3), we have the following equation

\[
t = \sin \varphi \sin \mu e_1 + \sin \varphi \cos \mu e_2 + \cos \varphi e_3. \quad (3.7)
\]

The covariant derivative of the vector field $t$ is:

\[
\nabla_t t = (t'_1 + t_2 t_3) e_1 + (t'_2 - t_1 t_3) e_2 + t'_3 e_3. \quad (3.8)
\]

From above equation we have

\[
\mu(\sigma) = \left(\frac{\sqrt{1 + \kappa^2} \sin \varphi}{\sin \varphi} - \cos \varphi \right) \sigma + M_1, \quad (3.10)
\]

where $M_1$ is a constant of integration.

Thus (3.9) and (3.10), imply

\[
t = \sin \varphi \sin[\left(\frac{\sqrt{1 + \kappa^2} \sin \varphi}{\sin \varphi} - \cos \varphi \right) \sigma + M_1] e_1
+ \sin \varphi \cos[\left(\frac{\sqrt{1 + \kappa^2} \sin \varphi}{\sin \varphi} - \cos \varphi \right) \sigma + M_1] e_2 + \cos \varphi e_3. \quad (3.11)
\]
Using (2.1) in (3.11), we obtain

\[
\mathbf{t} = (\sin \varphi \sin \left( \frac{\sqrt{1 + \kappa_2^2} \sin \varphi - \cos \varphi}{\sin \varphi} \right), \sin \varphi \cos \left( \frac{\sqrt{1 + \kappa_2^2} \sin \varphi - \cos \varphi}{\sin \varphi} \right), 
\cos \varphi + \sin \varphi \left( -\frac{\sin^2 \varphi}{\sqrt{1 + \kappa_2^2} - \sin \varphi \cos \varphi} \right) \cos \left( \frac{\sqrt{1 + \kappa_2^2} \sin \varphi - \cos \varphi}{\sin \varphi} \right) - \cos \varphi \sigma + M_1), \right)
\]

where \( M_1, M_2 \) are constants of integration.

Integrating both sides, we have (3.9). This proves our assertion. Thus, the proof of theorem is completed.

We can use Mathematica in above theorem, yields

\[
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