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A Note On The Generalized q-Genocchi measures with weight α

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ABSTRACT: In this paper we investigate special generalized q-Genocchi measures. We introduce q-Genocchi measures with weight α. The present paper deals with q-extension of Genocchi measure. Some earlier results of T. Kim in terms of q-Genocchi polynomials can be deduced. We apply the method of generating function, which are exploited to derive further classes of q-Genocchi polynomials and develop q-Genocchi measures. To be more precise, we present the integral representation of p-adic q-Genocchi measure with weight α which yields a deeper insight into the effectiveness of this type of generalizations. Generalized q-Genocchi numbers with weight α possess a number of interesting properties which we state in this paper.

Contents

1. Introduction, Definitions and Notations

The study of q-Genocchi measures and q-Genocchi polynomials and their combinatorial relations has received much attention [1], [2], [3], [5], [7], [9], [11], [12], [16]. Genocchi numbers and specially q-Genocchi numbers are the signs of very strong bond between elementary number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory(p-adic L-functions), quantum physics(quantum Groups). For showing the value of this type of numbers, we list some of their applications. One of applications of Genocchi numbers that was investigated by Jeff Remmel, In [24], is counting the number of up-down ascent sequences. Another application of Genocchi numbers is in Graph theory. For instance, Boolean numbers of the associated Ferrers Graphs are the Genocchi numbers of the second kind. Also one of application of q-Genocchi numbers is in q-Analysis. Actually there is an unexpected connection of the literature of p-adic analysis with q-analysis and quantum Groups, quantum top, and thus with non commutative Geometry, and q-analysis. For instance, spherical functions on quantum Groups are q-special functions. The q-Genocchi numbers can be defined in number of ways. The way in which it is defined is often determined by which sorts of applications they are intended to be

2000 *Mathematics Subject Classification:* 11R33 used for. There exist two important definitions of the q-Genocchi numbers and polynomials. The generating function definition, which is the most commonly used definition, and q-analog of Seidel's triangle associated to Genocchi numbers. As such it makes it very appealing for use in combinatorial applications. This type of definition have interesting combinatorial interpret ions in the classical model for q-Genocchi numbers. In the last decade, a surprising number of papers appeared proposing new generalizations of the Genocchi polynomials and q-Genocchi polynomials to real and complex variables or treating other topics related to q-Genocchi numbers. In [8], Carlitz defined q-extentions of Bernoulli numbers and polynomials and after several mathematicians developed his definition in terms of Genocchi numbers and defined q-Genocchi numbers and polynomials. In [25], After Y. Simsek by applying a derivative operator and the Mellin transformation for q-Genocchi numbers defined q-analogue of the Genocchi zeta function, q-analogue Hurwitz type Genocchi zeta function and q-Genocchi type l-function. By using this functions, he constructed p-adic interpolation functions which interpolate generalized q-Genocchi numbers at negative integers. Next, Professor T. Kim found some connections between q-Genocchi numbers and q-Volkenborn integral. In [12], Kim studied some families of multiple Genocchi numbers and polynomials. By using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, they constructed p-adic Genocchi numbers and polynomials of higher order. Mehmet Cenkci, et al, in [7], defined q-extentions of p-adic measures and obtained general systems of congruences, including Kummer-type congruences for q-Genocchi numbers. T.Kim, et al, in [22] [23], presented new concept of Bernoulli and Euler measure of weight α. In this paper we give another construction of q-Genocchi numbers and show the Integral representation of p-adic q-Genocchi measures with weight α.

Assume that p be a fixed odd prime number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote by the ring of integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.Also we denote $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|.|_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. When one talks of q-extensions, q is considered in many ways, e.g. as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$, If $q \in \mathbb{C}$ we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1-q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following notation

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

Where $\lim_{q\to 1} [x]_q = x$; cf. [1-6,8,10-23,25].

For a fixed positive integer d with $(d, f) = 1$, we set

$$X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

And

$$a+dp^{N}\mathbb{Z}_{p}=\left\{x\in X\mid x\equiv a\left(\operatorname{mod}dp^{N}\right)\right\},$$

Where $a\in\mathbb{Z}$ satisfies the condition $0\leq a<dp^{N}$.

It is known that

$$\mu_{q}\left(x+p^{N}\mathbb{Z}_{p}\right)=\frac{q^{x}}{\left[p^{N}\right]_{q}}$$

Is a distribution on X for $q\in\mathbb{C}_{p}$ with $\left|1-q\right|_{p}\leq1$.

Let $UD\left(\mathbb{Z}_{p}\right)$ be the set of uniformly differentiable function on $\mathbb{Z}_{p}$. We say that f is a uniformly differentiable function at a point $a\in\mathbb{Z}_{p}$, if the difference quotient

$$F_{f}\left(x,y\right)=\frac{f\left(x\right)-f\left(y\right)}{x-y}$$

has a limit $f^{\prime}\left(a\right)$ as $\left(x,y\right)\rightarrow\left(a,a\right)$ and denote this by $f\in UD\left(\mathbb{Z}_{p}\right)$. The p-adic q-integral of the function $f\in UD\left(\mathbb{Z}_{p}\right)$ is defined by

$$I_{q}\left(f\right)=\int_{\mathbb{Z}_{p}}f\left(x\right)d\mu_{q}\left(x\right)=\lim_{N\rightarrow\infty}\frac{1}{\left[p^{N}\right]_{q}}\sum_{x=0}^{p^{N}-1}f\left(x\right)q^{x}\tag{1.2}$$

The bosonic integral is considered as the bosonic limit $q\rightarrow1$, $I_{1}\left(f\right)=\lim_{q\rightarrow1}I_{q}\left(f\right)$. Similarly we have p-adic fermionic integration defined by T.Kim [17], on $\mathbb{Z}_{p}$ as follows

$$I_{-q}\left(f\right)=\lim_{q\rightarrow-q}I_{q}\left(f\right)=\int_{\mathbb{Z}_{p}}f\left(x\right)d\mu_{-q}\left(x\right)$$

Let $q\rightarrow1$, then we have p-adic fermionic integral on $\mathbb{Z}_{p}$ as follows

$$I_{-1}\left(f\right)=\lim_{q\rightarrow-1}I_{q}\left(f\right)=\lim_{N\rightarrow\infty}\sum_{x=0}^{p^{N}-1}f\left(x\right)\left(-1\right)^{x},$$

So by applying $f\left(x\right)=e^{tx}$, we get

$$t\int_{\mathbb{Z}_{p}}e^{xt}d\mu_{-1}\left(x\right)=\frac{2t}{e^{t}+1}=\sum_{n=0}^{\infty}G_{n}\frac{t^{n}}{n!}\tag{1.3}$$

Where G_{n} are Genocchi numbers. By using (1.3), we have

$$\int_{\mathbb{Z}_{p}}e^{tx}d\mu_{-1}\left(x\right)=\sum_{n=0}^{\infty}\frac{G_{n+1}}{n+1}\frac{t^{n}}{n!}$$

so from above, we obtain

$$\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_{p}}x^{n}d\mu_{-1}\left(x\right)\right)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left(\frac{G_{n+1}}{n+1}\right)\frac{t^{n}}{n!}$$

So by computing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation we get

$$\frac{G_{n+1}}{n+1} = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x)$$

In [3], q-extension of Genocchi numbers are defined by

$$G_{0,q} = 0,\ q(qG_q + 1)^n + G_{n,q} = \begin{cases} [2]_q\,, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.4}$$

with the usual convention about replacing $(G_q)^n$ by $G_{n,q}$.

The (h,q)-extension of Genocchi numbers are defined by

$$G_{0,q}^{(h)} = 0,\ q\left(qG_q^{(h)} + 1\right)^n + G_{n,q}^{(h)} = \begin{cases} [2]_q\,, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.5}$$

with the usual convention about replacing $\left(G_q^{(h)}\right)^n$ by $G_{n,q}^{(h)}$ (see [6]).

Recently, for $n \in \mathbb{N}^*$, *Araci et al.* are defined weighted q-Genocchi numbers by

$$\widetilde{G}_{0,q}^{(\alpha)} = 0,\ q^{1-\alpha}\left(q\widetilde{G}_q^{(\alpha)} + 1\right)^n + \widetilde{G}_{n,q}^{(\alpha)} = \begin{cases} [2]_q\,, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases} \tag{1.6}$$

with the usual convention about replacing $\left(\widetilde{G}_q^{(\alpha)}\right)^n$ by $\widetilde{G}_{n,q}^{(\alpha)}$.

From (1.1) and (1.6), the Witt's formula for the q-Genocchi numbers and polynomials with weight α are defined by

$$\frac{\widetilde{G}_{n,q}^{(\alpha)}}{n} = \int_{\mathbb{Z}_p} [y]_{q^\alpha}^{n-1} d\mu_{-q}(y)\,, \text{ where } n \in \mathbb{N}. \tag{1.7}$$

The q-Genocchi polynomials with weight α are also defined by

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = q^{-\alpha x}\sum_{k=0}^{n}\binom{n}{k}q^{\alpha k x}[x]_{q^\alpha}^{n-k}\widetilde{G}_{k,q}^{(\alpha)} \tag{1.8}$$

By (1.1),(1.7) and (1.8), we can derive the Witt's formula for $\widetilde{G}_{n,q}^{(\alpha)}(x)$ as follows:

$$\frac{\widetilde{G}_{n,q}^{(\alpha)}(x)}{n} = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^{n-1} d\mu_{-q}(y)\,, \text{ where } n \in \mathbb{N}. \tag{1.9}$$

For $n \in \mathbb{N}^*$ and $d \in \mathbb{N}$, the distribution relation for the q-Genocchi polynomials with weight α are known that

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^{n-1}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^a \widetilde{G}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right), \quad \text{(see [2] [5])} \tag{1.10}$$

Obviously, a special case of (1.10) when $\alpha = 1$ is $\lim_{q\to 1} \widetilde{G}_{n,q}^{(1)}(x) = G_n(x)$, where $G_n(x)$ are called Genocchi polynomials.

Let U be any compact open set of $\mathbb{Z}_p$, by the same method of [22], it can be written as a finite disjoint union of sets

$$U = \underset{j=1}{\overset{k}{\cup}} \left(a_j + p^N \mathbb{Z}_p\right),$$

where $N \in \mathbb{N}^*$ and $a_1, a_{2,...}, a_k \in \mathbb{N}$ with $0 \le a_i < p^N$.

Lemma 1.1. *Every map μ from the collection of compact-open sets in X to $\mathbb{Q}_p$ for which*

$$\mu\left(a + p^N \mathbb{Z}_p\right) = \underset{b=0}{\overset{p-1}{\cup}} \left(a + bp^N + dp^{N+1}\mathbb{Z}_p\right)$$

holds whenever $a+p^N\mathbb{Z}_p \subset X$, extends to a p-adic measure on X. (for proof see [22])

In this paper, we derive our results by using *Kim et al.* method in [22]. The purpose of this paper is to establish p-adic q-Genocchi measure with weight α on $\mathbb{Z}_p$ and to derive their integral represantations. Finally, we obtain generalized q-Genocchi numbers with weight α and some interesting properties of the generalized q-Genocchi numbers with weight α.

2. p-adic q-Genocchi measure with weight α

In this section, by using p-adic q-integral on $\mathbb{Z}_p$ and the following Eq. (2.1), we derive some interesting properties concerning the q-Genocchi numbers ans polynomials with weight α.

Now we define a map $\mu_{k,q}^{(\alpha)}$ on the balls in $\mathbb{Z}_p$ as follows:

$$\mu_{k,q}^{(\alpha)}\left(a + p^n \mathbb{Z}_p\right) = \frac{\left[p^n\right]_{q^\alpha}^{k-1}}{\left[p^n\right]_{-q}} (-1)^a\, q^a f_{k,p^n}^{(\alpha)}\left(\frac{\{a\}_n}{p^n}\right) \tag{2.1}$$

where $\{a\}_n = a \,(\mathrm{mod}\, p^n)$.

Theorem 2.1. *Let $\alpha, k \in \mathbb{N}$. Then we see that $\mu_{k,q}^{(\alpha)}$ is p-adic measure on $\mathbb{Z}_p$ if and only if*

$$\frac{[p]_{(q^{p^n})^\alpha}^{k-1}}{[p]_{(-q)^{p^n}}} \sum_{b=0}^{p-1} (-1)^{bp^n} q^{bp^n} f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{\frac{a}{p^n} + b}{p}\right) = f_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right).$$

Proof: For each $n \in \mathbb{N}$ and $0 \le a < p^n$. From expression (2.1),Then

$$\begin{aligned}\sum_{b=0}^{p-1} \mu_{k,q}^{(\alpha)}\left(a + bp^n + p^{n+1}\mathbb{Z}_p\right) &= \sum_{b=0}^{p-1} \frac{\left[p^{n+1}\right]_{q^\alpha}^{k-1}}{\left[p^{n+1}\right]_{-q}} (-q)^{a+bp^n} f_{k,q^{p^{n+1}}}^{(\alpha)}\left(\frac{a+bp^n}{p^{n+1}}\right) \\ &= (-1)^a\, q^a \frac{[p]_{(q^{p^n})^\alpha}^{k-1} \left[p^n\right]_{q^\alpha}^{k-1}}{[p]_{(-q)^{p^n}} \left[p^n\right]_{-q}} \sum_{b=0}^{p-1} (-1)^{bp^n} q^{bp^n} f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{\frac{a}{p^n}+b}{p}\right) \end{aligned} \tag{2.2}$$

By (2.1) , we note that $\mu_{k,q}^{(\alpha)}$ is p-adic measure on $\mathbb{Z}_p$ if and only if

$$f_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right)=\frac{[p]_{(q^{p^n})^\alpha}^{k-1}}{[p]_{(-q)^{p^n}}}\sum_{b=0}^{p-1}(-1)^{bp^n}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{\frac{a}{p^n}+b}{p}\right).$$

Thus, we complete the proof. □

Now, we set as follows:

$$f_{k,q^{p^n}}^{(\alpha)}(x)=\widetilde{G}_{k,q^{p^n}}^{(\alpha)}(x). \tag{2.3}$$

From (2.1) and (2.3), we simply see that

$$\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)=\frac{[p^n]_{q^\alpha}^{k-1}}{[p^n]_{-q}}(-1)^a q^a\widetilde{G}_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right) \tag{2.4}$$

By using (1.2), (1.10) and (2.4), we get the following theorem:

Theorem 2.2. *For $\alpha,k\in\mathbb{N}$,we have*

$$\int_X d\mu_{k,q}^{(\alpha)}(x)=\widetilde{G}_{k,q}^{(\alpha)}.$$

Proof: For each $k\in\mathbb{N}$, Thus

$$\begin{aligned}\int_X d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N\to\infty}\sum_{x=0}^{dp^N-1}\mu_{k,q}^{(\alpha)}\left(x+dp^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty}\frac{\left[dp^N\right]_{q^\alpha}^{k-1}}{[dp^N]_{-q}}\sum_{a=0}^{dp^N-1}(-1)^a q^a\widetilde{G}_{k,q^{dp^n}}^{(\alpha)}\left(\frac{a}{dp^N}\right)\\ &= \widetilde{G}_{k,q}^{(\alpha)}\end{aligned}$$

We arrive at the desired result. □

Let χ be Dirichlet's character with conductor $d\in\mathbb{N}$. Then we define the generalized q-Genocchi numbers attached to χ as follows:

$$\begin{aligned}\widetilde{G}_{n,\chi,q}^{(\alpha)} &= n\int_X\chi(x)[x]_{q^\alpha}^{n-1}d\mu_{-q}(x)\\ &= \lim_{N\to\infty}\frac{n}{[dp^N]_{-q}}\sum_{x=0}^{dp^N-1}(-1)^x q^x\chi(x)[x]_{q^\alpha}^{n-1}\\ &= \frac{[d]_{q^\alpha}^{n-1}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^a\chi(a)\left(n\lim_{N\to\infty}\frac{1}{[p^N]_{(-q)^d}}\sum_{x=0}^{p^N-1}\left(-q^d\right)^x\left[\frac{a}{d}+x\right]_{q^{\alpha d}}^{n-1}\right)\\ &= \frac{[d]_{q^\alpha}^{n-1}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^a\chi(a)\widetilde{G}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right). \end{aligned} \tag{2.5}$$

From (2.4) and (2.1), we can derive the following theorem.

Theorem 2.3. *For each $k \in \mathbb{N}$, we get*

$$\int_X \chi(x)\, d\mu_{k,q}^{(\alpha)}(x) = \widetilde{G}_{k,\chi,q}^{(\alpha)}.$$

Proof: For $k \in \mathbb{N}$ and by using (2.4) and (2.1), then

$$\begin{aligned}
\int_X \chi(x)\, d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} \chi(x)\, \mu_{k,q}^{(\alpha)}\left(x + dp^N \mathbb{Z}_p\right) \\
&= \lim_{N\to\infty} \frac{\left[dp^N\right]_{q^\alpha}^{k-1}}{\left[dp^N\right]_{-q}} \sum_{x=0}^{dp^N-1} \chi(x)(-1)^x q^x \widetilde{G}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{x}{dp^N}\right)
\end{aligned}$$

$$= \frac{[d]_{q^\alpha}^{k-1}}{[d]_{-q}} \sum_{a=0}^{d-1} (-1)^a q^a \chi(a) \lim_{N\to\infty} \frac{\left[p^N\right]_{q^{\alpha d}}^{k-1}}{\left[p^N\right]_{(-q)^d}} \sum_{x=0}^{p^N-1} (-1)^{dx} q^{dx} \widetilde{G}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{\frac{a}{d}+x}{p^N}\right)$$

$$= \frac{[d]_{q^\alpha}^{k-1}}{[d]_{-q}} \sum_{a=0}^{d-1} (-1)^a q^a \chi(a)\, \widetilde{G}_{k,q^d}^{(\alpha)}\left(\frac{a}{d}\right)$$

we obtain the desired result. □

Theorem 2.4. *For each $k \in \mathbb{N}$, we get*

$$\int_{pX} \chi(x)\, d\mu_{k,q}^{(\alpha)}(x) = \chi(p) \frac{[p]_{q^\alpha}^{k-1}}{[p]_{-q}} \widetilde{G}_{k,\chi,q^p}^{(\alpha)}.$$

Proof: From (2.4) and (2.1), Then

$$\int_{pX} \chi(x)\, d\mu_{k,q}^{(\alpha)}(x) = \lim_{N\to\infty} \frac{\left[dp^{N+1}\right]_{q^\alpha}^{k-1}}{\left[dp^{N+1}\right]_{-q}} \sum_{x=0}^{dp^N-1} \chi(px)(-1)^{px} q^{px} \widetilde{G}_{k,q^{dp^{N+1}}}^{(\alpha)}\left(\frac{px}{dp^{N+1}}\right)$$

$$= \frac{[p]_{q^\alpha}^{k-1}}{[p]_{-q}} \frac{[d]_{q^{\alpha p}}^{k-1}}{[d]_{(-q)^p}} \sum_{a=0}^{d-1} \chi(pa)(-q)^{pa} \lim_{N\to\infty} \frac{\left[p^N\right]_{q^{dp\alpha}}^{k-1}}{\left[p^N\right]_{(-q)^{dp}}} \sum_{x=0}^{p^N-1} (-q)^{dpx}\, \widetilde{G}_{k,q^{pdp^N}}^{(\alpha)}\left(\frac{dp\left(x+\frac{a}{d}\right)}{pdp^N}\right)$$

$$\begin{aligned}
&= \frac{[p]_{q^\alpha}^{k-1}}{[p]_{-q}} \frac{[d]_{q^{\alpha p}}^{k-1}}{[d]_{(-q)^p}} \sum_{a=0}^{d-1} (-1)^a \chi(p)\chi(a) q^{pa} \widetilde{G}_{k,q^{pd}}^{(\alpha)}\left(\frac{a}{d}\right) \\
&= \chi(p) \frac{[p]_{q^\alpha}^{k-1}}{[p]_{-q}} \widetilde{G}_{k,\chi,q^p}^{(\alpha)}.
\end{aligned}$$

Thus, we get the desired result. □

So, we get the following theorems with the same method of Theorem 2.2, Theorem 2.3 and Theorem 2.4

Theorem 2.5. *For $\beta\,(\neq 1) \in X^{*}$, we get*

$$\int_{pX} \chi\left(x\right) d\mu^{(\alpha)}_{k,q^{\frac{1}{\beta}}}\left(\beta x\right) = \chi\left(\frac{p}{\beta}\right) \frac{\left[p\right]^{k-1}_{q^{\frac{\alpha}{\beta}}}}{\left[p\right]_{(-q)^{\frac{1}{\beta}}}} \widetilde{G}^{(\alpha)}_{k,\chi,q^{\frac{p}{\beta}}}$$

Theorem 2.6. *For $\beta\,(\neq 1) \in X^{*}$, we get*

$$\int_{X} \chi\left(x\right) d\mu^{(\alpha)}_{k,q^{\frac{1}{\beta}}}\left(\beta x\right) = \chi\left(\frac{1}{\beta}\right) \widetilde{G}^{(\alpha)}_{k,\chi,q^{\frac{1}{\beta}}}$$

We can define the following equation,

$$\mu^{(\alpha)}_{k,\beta,q}\left(U\right) = \mu^{(\alpha)}_{k,q}\left(U\right) - \beta^{-1} \frac{\left[\beta^{-1}\right]^{k-1}_{q^{\alpha}}}{\left[\beta^{-1}\right]_{-q}} \mu_{k,q^{\frac{1}{\beta}}}\left(\beta U\right) \tag{2.6}$$

Theorem 2.7. *For $\beta\,(\neq 1) \in X^{*}$, we get*

$$\int_{X^{*}} \chi\left(x\right) d\mu^{(\alpha)}_{k,\beta,q}\left(\beta x\right) = \left(1-\chi^{p}\right)\left(1-\beta^{-1}\chi^{\beta^{-1}}\right) \widetilde{G}^{(\alpha)}_{k,\chi,q}.$$

Proof: By the definition of $\mu^{(\alpha)}_{k,\beta,q}$, from Theorem 2.4, Theorem 2.5 and Theorem 2.6, we obtain

$$\begin{aligned} &= \widetilde{G}^{(\alpha)}_{k,\chi,q} - \chi\left(p\right) \frac{\left[p\right]^{k-1}_{q^{\alpha}}}{\left[p\right]_{-q}} \widetilde{G}^{(\alpha)}_{k,\chi,q^{p}} - \frac{1}{\beta} \frac{\left[\frac{1}{\beta}\right]^{k-1}_{q^{\alpha}}}{\left[\frac{1}{\beta}\right]_{-q}} \chi\left(\frac{1}{\beta}\right) \widetilde{G}^{(\alpha)}_{k,\chi,q^{\frac{1}{\beta}}} \\ &\quad + \chi\left(\frac{p}{\beta}\right) \frac{\left[\frac{p}{\beta}\right]^{k-1}_{q^{\alpha}}}{\left[\frac{p}{\beta}\right]_{-q}} \widetilde{G}^{(\alpha)}_{k,\chi,q^{\frac{p}{\beta}}} \\ &= \left(1-\chi^{p}\right)\left(1-\beta^{-1}\chi^{\beta^{-1}}\right) \widetilde{G}^{(\alpha)}_{k,\chi,q} \end{aligned}$$

where the operator $\chi^{y} = \chi^{y,k,\alpha;q}$ on $f\left(q\right)$ defined by

$$\chi^{y} f\left(q\right) = \chi^{y,k,\alpha;q} f\left(q\right) = \frac{\left[y\right]^{k-1}_{q^{\alpha}}}{\left[y\right]_{-q}} \chi\left(y\right) f\left(q^{y}\right) \tag{2.7}$$

From expression (2.7), we get

$$\begin{aligned}
\chi^{x,k,\alpha;q} \circ \chi^{y,k,\alpha;q} f(q) &= \chi^{x,k,\alpha;q} \frac{[y]_{q^\alpha}^{k-1}}{[y]_{-q}} \chi(y) f(q^y) \\
&= \frac{[y]_{q^\alpha}^{k-1}}{[y]_{-q}} \chi(y) \chi(x) \frac{[y]_{q^{\alpha y}}^{k-1}}{[y]_{(-q)^y}} \chi(y) f(q^{xy}) \\
&= \frac{[xy]_{q^\alpha}^{k-1}}{[xy]_{-q}} \chi(xy) f(q^{xy}) \\
&= \chi^{xy,k,\alpha;q} f(q) \\
&= \chi^{xy} f(q).
\end{aligned}$$

Assume that define $\chi^x.\chi^y = \chi^{x,k,\alpha;q}.\chi^{y,k,\alpha;q}$. Then we have

$$\chi^x.\chi^y = \chi^{xy}.$$

By the definition χ^x, we can simply derive the following equation:

$$(1-\chi^p)\left(1-\beta^{-1}\chi^{\beta^{-1}}\right) = 1-\beta^{-1}\chi^{\beta^{-1}} - \chi^p + \beta^{-1}\chi^{p\beta^{-1}}$$

Assume that $f(q) = \widetilde{G}_{k,\chi,q}^{(\alpha)}$. Then we get

$$(1-\chi^p)\left(1-\beta^{-1}\chi^{\beta^{-1}}\right)\widetilde{G}_{k,\chi,q}^{(\alpha)}$$

Thus, we arrive at the desired result. □

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