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Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator

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ABSTRACT: In this paper, we establish the existence of at least three solutions to a boundary problem involving the p(x)-biharmonic operator. Our technical approach is based on theorem obtained by B. Ricceri's variational principle and local mountain pass theorem without (Palais-Smale) condition.

Key Words: p(x)-biharmonic, Neumann problem, variational methods.

Contents

1. Introduction

The study of various mathematical problems with variable exponent have received a lot of attention in recent years [1,14]. Fourth order equations appears in many contexts. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [9]). In addition, this type of equation can describe the static from change of beam or the sport of rigid body, there are many authors have pointed out that type of non linearity furnishes a model to study traveling waves in suspension bridges (see [5,11]).

In this paper, we consider the following $p(x)-$biharmonic problem with a boundary condition,

$$(\mathcal{P}) \quad \begin{cases} \Delta^2_{p(x)} u + a(x) \mid u \mid^{p(x)-2} u = f(x,u) + \lambda g(x,u) \ \ in \ \ \Omega, \\ Bu = Tu = 0 \ \ on \ \ \partial\Omega, \end{cases}$$

here Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)} u = \Delta(\mid \Delta u \mid^{p(x)-2} \Delta u)$ is the $p(x)-$biharmonic with $p \in C(\overline{\Omega})$, $p(x) > 1$, $\lambda \in \mathbb{R}$, $a \in L^\infty(\Omega)$ such that $\inf_{x\in\Omega} a(x) = a^- > 0$.
$Bu = Tu = 0$ denotes the following boundary conditions:
(1) $B = B_1, T = T_1$, Navier boundary condition, i.e. $B_1 u = \Delta u = 0$ and $T_1 u =$

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$u = 0$ *on* $\partial\Omega$.
(2) $B = B_2, T = T_2$, Neumann boundary condition, i.e $B_2u = \frac{\partial u}{\partial \nu} = 0$ and $T_2u = \frac{\partial}{\partial \nu}(| \Delta u |^{p(x)-2} \Delta u) = 0$ *on* $\partial\Omega$, where ν is the outward unit normal to $\partial\Omega$. Denote by $F(x,t) = \int_0^t f(x,s)ds$, $G(x,t) = \int_0^t g(x,s)ds$, $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Throughout this paper, we suppose the following assumptions:

$$(F) \quad f, g \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R}) \ \ such \ \ that \quad | f(x,t) |, | g(x,t) | \leq C_1 + C_2 | t |^{q(x)-1} \quad (1.1)$$

$\forall (x,t) \in \overline{\Omega} \times \mathbb{R}$, where $q \in C(\overline{\Omega}), C_1, C_2 > 0$ and $1 \leq q(x) < p^*(x) \ \ \forall x \in \overline{\Omega}$,
with

$$p^*(x) := \begin{cases} \frac{N-p(x)}{N-2p(x)} & if \ \ p(x) < \frac{N}{2}, \\ +\infty & if \ \ p(x) \geq \frac{N}{2}. \end{cases}$$

(F_1) $\lim_{|t|\to\infty}[F(x,t) - \frac{\lambda_1}{p(x)} | t |^{p^-}] = -\infty$ uniformly for almost every $x \in \overline{\Omega}$.
(F_2) There exist $x_0 \in \Omega, r_0 \in]0,1[$ and $t_0 > 1$ with $B(x_0, 2r_0) \subset \Omega$ such that
$F(x,t) \geq 0$ *for* $x \in B(x_0, 2r_0) \subset \Omega$ *and* $t \in]0, t_0]$,
$F(x, t_0) \geq C_0$ *for* $x \in B(x_0, r_0)$.
Where

$$C_0 = [(\frac{2}{r_0})^{p^+(B(x_0,2r_0))}(2^N - 1) + | a |_\infty 2^N] \frac{| t_0 |^{p^+(B(x_0,2r_0))}}{p^-(B(x_0, 2r_0))},$$

and

$$p^-(B(x_0, 2r_0)) = \inf_{x\in B(x_0,2r_0)} p(x), \ \ p^+(B(x_0, 2r_0)) = \sup_{x\in B(x_0,2r_0)} p(x).$$

(F_2') There exist $\xi \in \mathbb{R}$ such that

$$\int_\Omega F(x,\xi)dx > \int_\Omega \frac{a(x)}{p(x)} | \xi |^{p(x)} dx.$$

(F_3) There exist $b_0 > 0, \delta > 0$ such that
$F(x,t) \leq b_0 | t |^{q_0(x)}, \forall x \in \Omega, | t | < \delta$, where $q_0 \in C(\overline{\Omega})$ with $p^+ < q_0(x) < p^*(x)$ for $x \in \overline{\Omega}$.
(G_1) There exist an open ball $B(x_1, r_1) \subset \Omega, \beta \in C(B(x_1, r_1), \mathbb{R})$ with $1 \leq \beta(x) \leq \beta^+(B(x_1, r_1)) \leq p^-(B(x_1, r_1))$, $b > 0$ and $\gamma > 0$ such that

$$G(x,t) \geq b | t |^{\beta^+(B(x_1,r_1))} \quad for \ \ all \ \ x \in (B(x_1, r_1)) \ \ and \quad | t | < \gamma.$$

With

$$\beta^+(B(x_1, r_1)) = \sup_{x\in B(x_1,r_1)} \beta(x), \ \ \beta^-(B(x_1, r_1) = \inf_{x\in B(x_1,r_1)} \beta(x).$$

$$(G_2) \quad \limsup_{t\to 0} \frac{\inf_{x\in\Omega} G(x,t)}{| t |^{p^-}} = +\infty.$$

In the case $B = B_1$ and $T = T_1$, we claim the following theorem.

Theorem 1.1. *Suppose that assumptions* (F_1), (F_2), (F_3), (G_1) *and* (F) *hold. Then, there exists* $\lambda_* > 0$ *such that for any* $\lambda \in]0, \lambda_*[$, *the problem* $(\mathcal{P})$ *admits at least three weak solutions.*

The case $B = B_2, T = T_2$, we have the following result.

Theorem 1.2. *Under the assumptions* (F_1) ,(F_2') ,(F_3) (G_2) *and* (F). *Then, there exists* $\lambda_* > 0$ *such that for any* $\lambda \in]0, \lambda_*[$, *the problem* $(\mathcal{P})$ *has at least three weak solutions.*

Many authors consider the existence of multiple nontrivial solutions for some fourth order problems [11,16]. In particular, Li and Tang [10] consider the p-biharmonic equation. Using the modified three critical points theorem of B. Ricceri they get at least three solutions. The $p(x)-$biharmonic operator possesses more complicated nonlinearities than $p-$biharmonic, for example, it is inhomogeneous. Recently, in [4] A. Ayoujil and A. R. El Amrouss interested to the spectrum of a fourth order elliptic equation with variable exponent. They proved the existence of infinitely many eigenvalue sequences and $sup\Lambda = +\infty$, where Λ is the set of all eigenvalues. Moreover, they present some sufficient conditions for $inf\Lambda = 0$.

The technical approach is based on the Ricceri's variational principle and local mountain pass theorem [3], without Palais- Smale condition. One of the first result in this direction was obtained by Shao-Gao Deng [6] for the $p(x) - laplacien$, here, we borrow some ideas from that work. The purpose of this work is to improve the results of [6] and extend them to the case of p(x)-biharmonic equation with Navier and Neumann condition.

This article consists of three sections. In section 2, we start with some preliminary basic results on theory of Lesbegue-Sobolev spaces with variables exponent (we refer to the book of Musielak [13] , Mihăilescu and Rădulescu [12]), we recall Ricceri's variational principle with some results which are needed later. In section 3, we give the proof of the main result.

2. Preliminaries

2.1. Variable exponent space and Sobolev Spaces

In order to deal with the problem $(\mathcal{P})$, we need some theory of variable exponent Sobolev Space. For convenience, we only recall some basic facts which will be used later. Suppose that $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Let $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \ \ such\ that \ \ \inf_{x\in\overline{\Omega}} p(x) > 1\}$. For any $p(x) \in C_+(\overline{\Omega})$, denote by $p_k^*(x) = \begin{cases} \frac{N-p(x)}{N-kp(x)} & if \ \ kp(x) < N, \\ +\infty & if \ \ kp(x) \geq N. \end{cases}$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$,
$L^{p(x)}(\Omega) = \{u \in L^1(\Omega) : \ \int_\Omega \mid u \mid^{p(x)} dx < \infty\}$ then $L^{p(x)}(\Omega)$ endowed with the norm

$$\| u \|_{p(x)} = \inf\{\lambda > 0 : \ \int_\Omega \mid \frac{u}{\lambda} \mid^{p(x)} dx \leq 1\}$$

becomes a Banach space separable and reflexive space.

Proposition 2.1. *Set,* $\rho(u) = \int_\Omega \mid u \mid^{p(x)} dx$, *if* $u \in L^{p(x)}(\Omega)$ *we have :*
(1) $\| u \|_{p(x)} \geq 1 \Rightarrow \| u \|_{p(x)}^{p^-} \leq \rho(u) \leq \| u \|_{p(x)}^{p^+}$.
(2) $\| u \|_{p(x)} \leq 1 \Rightarrow \| u \|_{p(x)}^{p^-} \geq \rho(u) \geq \| u \|_{p(x)}^{p^+}$.

Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), \mid \alpha \mid \leq k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index and $\mid \alpha \mid = \Sigma_{i=1}^N \alpha_i$.
The space $W^{k,p(x)}(\Omega)$ with the norm $\| u \| = \Sigma_{|\alpha \leq k|} \| D^\alpha u \|_{p(x)}$ is a Banach separable and reflexive space.

Proposition 2.2. *([2,7,13]). For* $p, r \in C_+(\overline{\Omega})$ *such that* $r(x) \leq p_k^*(x)$ *for all* $x \in \overline{\Omega}$, *there is a continuous and compact embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

We denote by $W_0^{k,p(x)}(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p(x)}(\Omega)$.

Remark 2.1. *1)*$(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \| . \|)$ *is a Banach space separable and reflexive space.*
2) Define $\| u \|_a = \inf\{\lambda > 0 : \int_\Omega [\mid \frac{\Delta u}{\lambda} \mid^{p(x)} + a(x) \mid \frac{u}{\lambda} \mid^{p(x)}] dx \leq 1\}$, *for all* $u \in W^{2,p(x)}(\Omega)$ *or* $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ *then* $\| u \|_a$ *is a norm on* $W^{2,p(x)}(\Omega)$ *and* $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, *equivalent the usual one.*
3) By the above remark and proposition 2.2 there is a continuous and compact embedding of $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ *into* $L^{r(x)}(\Omega)$, *where* $r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$.

Proposition 2.3. *Set* $\rho_a(u) = \int_\Omega [\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)}] dx$. *For* $u, u_n \in W^{2,p(x)}(\Omega)$ *we have,*
(1) $\| u \|_a \leq 1 \Rightarrow \| u \|_a^{p^+} \leq \rho_a(u) \leq \| u \|_a^{p^-}$.
(2)$\| u \|_a \geq 1 \Rightarrow \| u \|_a^{p^+} \geq \rho_a(u) \geq \| u \|_a^{p^-}$.
(3) $\| u_n \|_a \to 0 \Leftrightarrow \rho_a(u_n) \to 0$.
(4) $\| u_n \|_a \to +\infty \Leftrightarrow \rho_a(u_n) \to +\infty$.

The proof is similar to proof in [7] [Theorem 3.1].

Proposition 2.4. *([7]). For any* $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$, *we have*

$$\mid \int_\Omega uv dx \mid \leq (\frac{1}{p^-} + \frac{1}{q^-}) \| u \|_{p(x)} \| v \|_{q(x)},$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

We denote by $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ when $B = B_1, T = T_1$ and $X = W^{2,p(x)}(\Omega)$ if $B = B_2$ and $T = T_2$.

Definition 2.1. *Let $u \in X$, u is called* ***weak solution*** *of problem* $(\mathcal{P})$ *if for all* $v \in X$,

$$\int_\Omega \mid \Delta u \mid^{p(x)-2} \Delta u \Delta v dx + \int_\Omega a(x) \mid u \mid^{p(x-2)} uv dx = \int_\Omega f(x,u) v dx + \lambda \int_\Omega g(x,u) v dx.$$

We define ,
$I(u) = \int_\Omega \frac{1}{p(x)}[\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)}]dx - \int_\Omega F(x,u)dx$ and $J(u) = -\int_\Omega G(x,u)dx$,
where $F(x,t) = \int_0^t f(x,s)ds$, $G(x,t) = \int_0^t g(x,s)ds$ and $\varphi = I + \lambda J, \lambda \in \mathbb{R}$.

The following proposition will be used later,

Proposition 2.5. *(1) Let $L(u) = \int_\Omega \frac{1}{p(x)}[\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)}]dx$. Then the functional $L : X \to \mathbb{R}$ is sequentially weakly lower semi continuous, $L \in C^1(X, \mathbb{R})$.*
(2) the mapping $L' : X \to X'$ is a strictly monotone, bounded homeomorphism and is of type S_+ , namely, $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} L'(u_n)(u_n - u) \leq 0$ implies that $u_n \to u$, where $\to$ and $\rightharpoonup$ denote the strong and weak convergence respectively.

Proof: : (1) Since $\int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx \in C^1(X, \mathbb{R})$ then L is well defined and $L \in C^1(X, \mathbb{R})$. By the continuity and convexity of L, we deduce that L is sequentially weakly lower semi continuous .
(2) Since L' is Fréchet derivative of L then L is continuous and bounded. We set

$$U_p = \{x \in \Omega : p(x) \geq 2\},$$

$$V_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

By the elementary inequalities, we have $\forall x, y \in \mathbb{R}^N$

$$\mid x - y \mid^\gamma \leq 2^\gamma (\mid x \mid^{\gamma-2} x - \mid y \mid^{\gamma-2} y).(x-y) \quad if \quad \gamma \geq 2,$$

$$\mid x - y \mid^2 \leq \frac{1}{\gamma - 1}(\mid x \mid + \mid y \mid)^{2-\gamma}(\mid x \mid^{\gamma-2} x - \mid y \mid^{\gamma-2} y).(x-y) \quad if \quad 1 < \gamma < 2,$$

where $x.y$ denotes the usual inner product in $\mathbb{R}^N$. Then for all $u, v \in X$ such that $u \neq v$ $\langle L'(u) - L'(v), u - v \rangle > 0$, which means that L' is strictly monotone. Let $(u_n)_n$ be a sequence of X such that

$$u_n \rightharpoonup u \quad \text{in} \quad X$$

and

$$\limsup_{n\to\infty} \langle L'(u_n), u_n - u \rangle \leq 0.$$

It suffices to show that

$$\int_\Omega (\mid \Delta u_n - \Delta u \mid^{p(x)} + a(x) \mid u_n - u \mid^{p(x)}) \to 0, \tag{2.1}$$

by the monotonicity of L' , we claim that

$$\langle L'(u_n) - L'(u), u_n - u\rangle \geq 0.$$

Since $u_n \rightharpoonup u$ in X then

$$\limsup_{n\to\infty}\langle L(u_n) - L(u), u_n - u\rangle = 0. \tag{2.2}$$

Put

$$\chi_n(x) = (\mid \Delta u \mid^{p(x)-2} \Delta u_n - \mid \Delta u \mid^{p(x)-2} \Delta u).(\Delta u_n - \Delta u),$$
$$\xi_n(x) = (\mid u_n \mid^{p(x)-2} u_n - \mid u \mid^{p(x)-2} u).(u_n - u).$$

By the compact embedding of X into $L^{p(x)}(\Omega)$,

$$u_n \to u \;\; in \;\; L^{p(x)}(\Omega),$$
$$\mid u_n \mid^{p(x)-2} u_n \to \mid u \mid^{p(x)-2} u \;\; in \;\; L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in \Omega$. It follows that,

$$\int_\Omega \xi_n(x)dx \to 0, \tag{2.3}$$

using (2.1) and (3.3), then we have

$$\limsup_{n\to\infty}\int_\Omega \chi_n(x)dx = 0. \tag{2.4}$$

Thanks to the above inequalities,

$$\int_{U_p} \mid \Delta u_n - \Delta u_k \mid^{p(x)} dx \leq 2^{p^+}\int_{U_p}\chi_n(x)dx,$$
$$\int_{U_p} \mid u_n - u_k \mid^{p(x)} dx \leq 2^{p^+}\int_{U_p}\xi_n(x)dx.$$

It results that

$$\int_{U_p}(\mid \Delta u_n - \Delta u \mid) \mid^{p(x)} + a(x) \mid u_n - u \mid^{p(x)} dx \to 0 \;\; as \;\; n \to \infty. \tag{2.5}$$

Besides, in V_p, put $\delta_n = \mid \Delta u_n \mid + \mid \Delta u \mid$, we have

$$\int_{V_p} \mid \Delta u_n - \Delta u \mid^{p(x)} dx \leq \frac{1}{p^{-1}-1}\int_{V_p}(\chi_n)^{\frac{p(x)}{2}}(\delta_n)^{\frac{p(x)}{2}(2-p(x))}dx.$$

By Young's inequality,

$$\begin{aligned} d\int_{V_p} \mid \Delta u_n - \Delta u \mid^{p(x)} dx &\leq \int_{V_p}(d\chi_n)^{\frac{p(x)}{2}}(\delta_n)^{\frac{p(x)}{2}(2-p(x))}dx \\ &\leq \int_{V_p}\chi_n(d)^{\frac{2}{p(x)}}dx + \int_{V_p}(\delta_n)^{p(x)}dx. \end{aligned} \tag{2.6}$$

From (2.4) and since $\chi_n \geq 0$, one consider that

$$0 \leq \int_{V_p} \chi_n dx < 1.$$

If $\int_{V_p} \chi_n dx = 0$ then $\int_{V_p} \mid \Delta u_n - \Delta u \mid^{p(x)} dx = 0$. Or else, we take $d = (\int_{V_p} \chi_n dx)^{\frac{-1}{2}} > 1$ and the fact that $\frac{2}{p(x)} < 2$, inequality (2.6) becomes

$$\begin{aligned} \int_{V_p} \mid \Delta u_n - \Delta u \mid^{p(x)} dx &\leq \frac{1}{d}(\int_{V_p} \chi_n d^2 dx + \int_{\Omega} \delta_n^{p(x)} dx), \\ &\leq \int_{V_p} (\chi_n dx)^{\frac{1}{2}} (1 + \int_{\Omega} \delta_n^{p(x)} dx). \end{aligned}$$

Note that, $\int_{\Omega} \delta_n^{p(x)} dx$ is bounded, which implies

$$\int_{V_p} \mid \Delta u_n - \Delta u \mid^{p(x)} dx \to 0 \quad as \quad n \to \infty.$$

Similarly we can have

$$\int_{V_p} \mid u_n - u \mid^{p(x)} dx \to 0 \quad as \quad n \to \infty.$$

Hence, it result that

$$\int_{V_p} (\mid \Delta u_n - \Delta u \mid^{p(x)} + a(x) \mid u_n - u \mid^{p(x)}) dx \to 0 \quad as \quad n \to \infty. \tag{2.7}$$

Finally, (2.1) is given by combining (2.5) and (2.7). It remains to show that L' is is a homeomorphism. In view of strict monotonicity of L' which implies the injectivity of L'. Moreover, L' is a coercive. Indeed, since $p^{-1} > 1$, for each $u \in X$ such that $u \geq 1$ we have

$$\frac{\langle L'(u), u \rangle}{\| u \|} = \frac{\rho_a(u)}{\| u \|} \geq \| u \|^{p^- - 1} \to \infty \quad as \quad n \to \infty.$$

Consequently, thanks to a Minty-Browder [15], L' is surjective and admits an inverse mapping. It suffices to show the continuity of $(L')^{-1}$. Let $(f_n)_n$ be a sequence of X' such that $f_n \to f$ in X'. Let u_n and u in X such that

$$(L')^{-1}(f_n) = u_n \quad and \quad (L')^{-1}(f) = u.$$

By the coercivity of L', one deducts that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ in X, which implies

$$\lim_{n \to \infty} \langle L'(u_n) - L'(u), u_n - \widehat{u} \rangle = \lim_{n \to \infty} \langle f_n - f, u_n - \widehat{u} \rangle = 0.$$

Since L' is of (S_+) type and continuous, it follows that

$$u_n \to \widehat{u} \;\; in \;\; X \;\; and \;\; L'(u_n) \to L'(\widehat{u}) = L(u) \;\; in \;\; X'.$$

Moreover, since L' is an injection, we deduce that $u = \widehat{u}$. □

Proposition 2.6. *let $\sigma(u) = \int_\Omega G(x,u)dx$, then σ is a C^1 in $L^{q(x)}(\Omega)$ and σ' are weakly-strongly continuous, i.e $u_n \rightharpoonup u$ implies $\sigma'(u_n) \to \sigma'(u)$.*

Proof: : by (1.1) we have, $G(x,t) \leq A(x) + B \mid t \mid^{q(x)}$, where $A \in L^1(\Omega)$, $A(x) \geq 0$, $B \geq 0$. Then Nemytskii operator properties implies that σ is a C^1 in $L^{q(x)}(\Omega)$. By the continuous embedding of X into $L^{q(x)}(\Omega)$, we have σ is also C^1 in X and for $u, v \in X$

$$\sigma'(u)v = \int_\Omega g(x,u)v dx.$$

Let $(u_n)_n \subset X$ be a sequence such that $u_n \rightharpoonup u$. Since there is a compact embedding of X into $L^{q(x)}(\Omega)$, there is a subsequence, noted also $(u_n)_n$, such that $u_n \to u$ in $L^{q(x)}(\Omega)$. According to the Krasnoselki's theorem, the Nemytskii operator

$$\begin{array}{rlll} N_g: & L^{q(x)}(\Omega) & \to & L^{\frac{q(x)}{q(x)-1}}(\Omega) \\ & u & \mapsto & f(., \; u) \end{array}$$

is continuous. Hence, $N_g(u_n) \to N_g(u)$ in $L^{\frac{q(x)}{q(x)-1}}(\Omega)$. Using Holder's inequality and the continuous embedding of X into $L^{q(x)}(\Omega)$, we obtain

$$\begin{aligned} \mid \langle \sigma'(u_n) - \sigma'(u), v \rangle \mid &= \mid \int_\Omega (g(x,u_n) - g(x,u))v(x)dx \mid \\ &\leq 2 \parallel N_g(u_n) - N_g(u) \parallel_{\frac{q(x)}{q(x)-1}} \parallel v(x) \parallel_{q(x)} \\ &\leq C \parallel N_g(u_n) - N_g(u) \parallel_{\frac{q(x)}{q(x)-1}} \parallel v \parallel_a . \end{aligned}$$

Thus, $\sigma'(u_n) \to \sigma'(u)$.

□

2.2. Ricceri's variational principle

Definition 2.2. *Let G a bounded subset of X and $\rho \in \mathbb{R}$. G is called a **block** of I with type ρ if $I(u) < \rho \; \forall x \in G$ and $I(x) = \rho \; \forall x \in \overline{\partial G}$. Where $\overline{\partial G} = \overline{G}^W \backslash G$ and $\overline{G}^W$ is the closure of G in X in the weak topology.*

Definition 2.3. *Let D a bounded open subset of X and $c < b$ is called **Ricceri box** of I with the type (c, d) if*

$$c = \inf_D I < \inf_{\overline{\partial D}} I = b.$$

Definition 2.4. *Let Y be a Banach space, G_0 and G be two bounded open subset of Y with $\overline{G_0} \subset G$ and $\phi : Y \to \mathbb{R}$ a functional. (G_0, G) is a **valley box** of ϕ if*

$$\sup_{G_0} \phi < \inf_{\partial G} \phi.$$

Theorem 2.1. *(see [6,8]) Assume that $I,\ J : X \to \mathbb{R}$ are sequentially weakly lower semi continuous and G is a Ricceri block of I with type ρ. Let*

$$\lambda_* = \sup_{x\in G} \frac{\rho - I(x)}{J(x) - \inf_{\overline{G}^W} J}$$

then for each $\lambda \in]0, \lambda_*[$, the restriction of $I + \lambda J$ to $\overline{G}^W$ achieves its infimum at some $x_* \in G$, so x_* is a local minimizer of $I + \lambda J$.

Remark 2.2. *i) let $u_* \in X$ a strictly local minimizer of I, then for $\epsilon > 0$ small enough, we have $\inf_{\partial B(u,\epsilon)} I > I(u_*)$ i.e $B(u_*, \epsilon)$ is a Recceri box of I.*
ii) In fact, by proposition 2.6) in [8], $I,\ J : X \to \mathbb{R}$ are sequentially weakly lower semi continuous.

Proposition 2.7. *[8] Suppose that G is a Ricceri box of I with type (c, b) and $I : X \to \mathbb{R}$ continuous. Then for every $\rho \in]c, b]$ we have $I^{-1}(] - \infty, \rho[) \cap G$ is a Ricceri block of I with type ρ.*

By Proposition 2.5, Remark 2.2 and Theorem 2.1 we obtain the following result,

Proposition 2.8. *[6] Suppose that $I, J : X \to \mathbb{R}$ are continuous. For some $r > 0$, $u_1 \in B(u_0, r)$, $I(u_0) = \inf_{B(u_0,r)} I = c_0$; $\inf_{\partial B(x_0,r)} I = b > c_0$ and u_1 is a strictly local minimizer of I and $I(u_1) = c_1 > c_0$. Then for $\epsilon > 0$ small enough and $\rho_1 > c_1, \rho_0 \in]c_0, \min\{b, c_1\}[$ and $\forall\lambda \in]0, \lambda_*[$, $I + \lambda J$ has at least two local minima u_0^*, u_1^* in $B(u_0, r)$. Where $u_0^* \in I^{-1}(] - \infty, \rho_0[) \cap B(u_0, r), u_0^* \notin \overline{B(u_1, \epsilon)}$ and $u_1^* \in I^{-1}(] - \infty, \rho_1[) \cap B(u_1, r)$.*

Theorem 2.2. *[6] Let Y be a reflexive Banach space. Assume that*
1) $\phi \in C^1(Y, \mathbb{R})$, the mapping $\phi' : Y \to Y^*$ is of type S_+.
2) (G_0, G) is a valley box of ϕ with G_0, G being connected and $0 \in G_0$.
3) There exist $e \in G_0$ and $r > 0$ such that

$$\| e \| > r, \quad \inf_{\partial B(0,r)} \phi > \max\{\phi(0), \phi(e)\}.$$

Then, the functional ϕ has at least a critical point $u_0 \in \overline{G}$ with $\phi(u_0) = c$, where $c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \phi(\gamma(t))$ and $\Gamma = \{\gamma \in C([0, 1], G) : \gamma(0) = 0; \gamma(1) = e\}$.

Corollary 2.1. *Under the same assumption as in previous theorem, furthermore, if $J : Y \to \mathbb{R} \in C^1$ and $J' : Y \to Y'$ are weakly strongly continuous. Then, for each $\lambda \in]0, \lambda_*[$, $I + \lambda J$ has still a mountain pass type critical point $u_2 \in \overline{G}$.*

3. Proof of the main result

The critical point of the integral functional $\varphi = I + \lambda J$ is solution of the problem $(\mathcal{P})$. Define

$$\lambda_1 = \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}[\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)}]dx}{\int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx},$$

Proof: [proof of Theorem 1.1] The proof is divided into four steps,

`step(1):`

We show that $v_0 = 0$ is strictly local minimizer of I. By (1.1) and assumption (F_3), we may find $q_1 \in C(\overline{\Omega})$ with $p^+ < q_1^- \leq q_1(x) < p^*(x)$ such that

$$F(x,t) \leq c_3 \mid t \mid^{q_1(x)}, \quad \forall x \in \Omega, \ \forall t \in \mathbb{R}. \tag{3.1}$$

We can assume that $\parallel u \parallel_a < 1$ is small enough, thus

$$\begin{aligned} I(u) &\geq \int_\Omega \frac{1}{p(x)}[\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)} dx] - c_3 \int_\Omega \mid u \mid^{q_1(x)} \\ &\geq \frac{1}{p^+} \parallel u \parallel_a^{p^+} - c_4 \parallel u \parallel_a^{q_1^-}. \end{aligned}$$

Since $q_1^- > p^+$ then there exists $\epsilon > 0$ such that $\forall u \in \overline{B(0,\epsilon)} \setminus 0$ we have $I(u) > 0 = I(v_0)$.

`step(2):`

We show that the functional I has a global minimizer $v_1 \neq 0$. Set $H(x,t) = F(x,t) - \frac{\lambda_1}{p(x)} \mid t \mid^{p^-}$. Then, from (F_1) we conclude that, for every $M > 0$, there is $R_M > 0$ such that

$$H(x,t) \leq -M, \quad \forall \mid t \mid \geq R_M, \quad \text{almost every } x \in \Omega. \tag{3.2}$$

We have I is coercive, or else there exist $K \in \mathbb{R}$ and $(u)_n \subset X$ such that

$$\parallel u_n \parallel_a \to \infty \quad and \quad I(u_n) \leq K.$$

Putting $v_n = \frac{u_n}{\|u_n\|_a}$ i.e $\parallel v_n \parallel_a = 1$. Then for subsequence, we may assume that for $v \in X$, we have $v_n \rightharpoonup v$ in X, $v_n \to v$ strongly in $L^{p(x)}(\Omega)$, $v_n(x) \to v(x)$ for almost every $x \in \Omega$. Now, using 3.2, we obtain

$$\begin{aligned} &K \geq I(u_n) = \int_\Omega \frac{1}{p(x)}[\mid \Delta u \mid^{p(x)} + a(x) \mid u_n \mid^{p(x)}]dx - \int_\Omega F(x,u_n)dx. \\ &\geq \int_\Omega \frac{1}{p(x)}[\mid \Delta u_n \mid^{p(x)} + a(x) \mid u_n \mid^{p(x)}]dx - \lambda_1 \int_\Omega \frac{1}{p(x)} \mid u_n \mid^{p^-} dx - \int_\Omega H(x,u_n)dx \\ &\geq \frac{1}{p^+} \parallel u_n \parallel_a^{p^-} - \lambda_1 \int_\Omega \frac{1}{p(x)} \mid u_n \mid^{p^-} dx + M_1, \end{aligned} \tag{3.3}$$

where $M_1 \in \mathbb{R}$. Dividing 3.3 by $\parallel u_n \parallel_a^{p^-}$ and passing to the limit, we conclude

$$\frac{1}{p^+} - \lambda_1 \int_\Omega \mid v \mid^{p^-} dx \leq 0,$$

hence, $v \not\equiv 0$. Therefore $\mid \Omega^{`} \mid > 0$ such that $\Omega^{`} = \{x \in \Omega \setminus v_0(x) \neq 0\}$, then $\mid u_n(x) \mid \to +\infty$ for almost every $x \in \Omega^{`}$. On the other hand,

$$\begin{aligned} \lambda_1 \int_{[|u|\geq 1]} \frac{\mid u \mid^{p^-}}{p(x)} dx &\leq \lambda_1 \int_{[|u|\geq 1]} \mid u \mid^{p(x)} dx \\ &\leq \lambda_1 \int_{\Omega} \mid u \mid^{p(x)} dx \\ &\leq \int_{\Omega} \frac{1}{p(x)} [\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)}] dx. \end{aligned} \tag{3.4}$$

Where,

$$[\mid u \mid \geq 1] = \{x \in \Omega \setminus \mid u \mid \geq 1\} \;\; ; \;\; [\mid u \mid < 1] = \{x \in \Omega \setminus \mid u \mid < 1\}.$$

It is clear that $\int_{[|u_n|<1]} \frac{1}{p(x)} \mid u_n \mid^{p(x)} dx$ is bounded. From (F_1) and the above inequalities 3.4 we deduce

$$\begin{aligned} K &\geq \int_{\Omega} \frac{1}{p(x)} [\mid \Delta u \mid^{p(x)} + a(x) \mid u \mid^{p(x)}] dx - \int_{\Omega} F(x, u_n) dx \\ &= \int_{\Omega} \frac{1}{p(x)} [\mid \Delta u_n \mid^{p(x)} + a(x) \mid u_n \mid^{p(x)}] dx - \lambda_1 \int_{\Omega} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{\Omega} H(x, u_n) dx \\ &= \int_{\Omega} \frac{1}{p(x)} [\mid \Delta u_n \mid^{p(x)} + a(x) \mid u_n \mid^{p(x)}] dx - \lambda_1 \int_{[|u_n|\geq 1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx \\ &\quad -\lambda_1 \int_{[|u_n|<1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{\Omega} H(x, u_n) dx \\ &\geq \lambda_1 \int_{[|u_n|<1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{\Omega^{`}} H(x, u_n) dx + \int_{\Omega \setminus \Omega^{`}} H(x, u_n) dx \to +\infty, \end{aligned}$$

which is a contradiction. Hence I is coercive and has a global minimizer v_1. When the assumption (F_2) holds, taking $\omega_1 \in C_0^{\infty}(B(x_0, 2r_0))$ such that $0 \leq \omega_1 \leq t_0$ for all $x \in B(x_0, 2r_0)$, $\omega_1(x) \equiv t_0$ for $x \in B(x_0, r_0)$ and $\mid \Delta\omega_1(x) \mid \leq \frac{2t_0}{r_0}$ then $\omega_1 \in X$. On the other hand,

$$\begin{aligned} I(\omega_1) &\leq \int_{B(x_0,2r_0)\setminus B(x_0,r_0)} \mid \frac{2t_0}{r_0} \mid dx + \mid a \mid_{\infty} \int_{B(x_0,2r_0)} \frac{1}{p(x)} \mid t_0 \mid^{p(x)} dx - \int_{\Omega} F(x, \omega_1) dx \\ &\leq C_0 \mid B(x_0, r_0) \mid - \int_{\Omega} F(x, \omega_1).dx \end{aligned}$$

Since

$$\int_{\Omega} F(x, \omega_1) dx > \int_{B(x_0,r_0)} F(x, \omega_1) dx \geq C_0 \mid B(x_0, r_0) \mid,$$

we obtain $I(\omega_1) < 0$ then $I(v_1) < 0 = I(v_0)$. So $v_1 \neq 0$.

`step(3):`

We show that φ has two local minima. Since I is coercive there is $r_0 > 0$ large

enough such that $v_0, v_1 \in B(0, r_0)$ and $\inf_{\partial B(0,r_0)} I > I(v_0) > I(v_1)$. By proposition 2.8, given any $\epsilon > 0$, $\rho_1 \in]I(v_1), 0[$ and $\rho_2 > 0$ then $\forall \lambda \in]0, \lambda_*[$, φ has at least two local minima $u_0 \in B(0,\epsilon) \cap I^{-1}(]-\infty, \rho_2[)$, $u_1 \in I^{-1}(]-\infty, \rho_1[)$ and $u_1 \notin \overline{B(0,\epsilon)}$.
The minimizer $u_0 \neq 0$. In fact, when (G_1) holds, taking $\omega \in C_0^\infty(B(x_1, r_1))$ such that $0 \leq \omega \leq 1$ and $\omega(x) \equiv 1$ for $x \in B(x_1, \frac{r_1}{2})$, then, it is easy to see that for $\lambda \in]0, \lambda_*[$, when $t > 0$ is small enough, we get $t\omega \in B(0,\epsilon) \cap I^{-}(]-\infty, \rho_2[)$ and $I(u_0) + \lambda J(u_0) \leq I(t\omega) + \lambda J(t\omega) < 0$. In particular, $u_0 \neq 0$.

`step(4):`
φ has a mountain pass type critical point $\forall \lambda \in]0, \lambda_*[$. We take $r_1 > 0$ such that $B(0, r_1) \subset X$ and $B(0, r_1) \supset I^{-1}(]-\infty, \rho_1[) \cup B(0,\epsilon)$. Since I is coercive, there exists $r_2 > r_1$ such that

$$\inf_{\partial B(0,r_2)} I > \sup_{B(0,r_1)} I,$$

then $(B(0, r_1), B(0, r_2))$ is a valley box of I. Since $I(v_1) < 0 = I(v_0)$ and by step 1), we have that for some $\epsilon_0 > 0$ with $\epsilon_0 > \| v_1 \|_a$, $\inf_{\partial B(\epsilon,0)} I > 0$. We apply the Corollary 3.1, then φ admits a mountain pass point u_2. Consequently, u_0, u_1 and u_2 are at least three nontrivial solutions of the problem $(\mathcal{P})$. □

Proof: [proof of Theorem 1.2] It was the same steps of the previous proof.

`step(1):`
To show that $v_0 = 0$ is strictly local minimizer of I, we follow the same procedure as in step (1) in the previous proof.

`step(2):`
We show that the functional I has a global minimizer $v_1 \neq 0$. Similarly in step(2) in the last proof of Theorem 1.2, one shows the coercivity of I and then I has a global minimizer v_1. We use the condition (F_2'), we obtain $I(\xi) < 0$ and then $I(v_1) < 0 = I(v_0)$. So $v_1 \neq 0$.

`step(3):`
We show that φ has two local minima. The same way as In step 3 in the last proof of Theorem 1.2, φ has at least two local minima u_0 and $u_1 \neq 0$. Moreover, the minimizer $u_0 \neq 0$. Indeed, by assumption (G_2), taking $t_n \to 0$ such that

$$\frac{\inf_{x\in\Omega} G(x, t_n)}{| t_n |^{p^-}} \to +\infty.$$

Let $w_n = t_n($ i.e $w_n(x) \in X)$. For all $\lambda \in]0, \lambda_*[$, by 3.1, we get

$$\begin{aligned}
\varphi(w_n) &\leq | t_n |^{p^-} \int_\Omega \frac{a(x)}{p(x)} dx - \int_\Omega F(x, t_n) dx - \lambda \int_\Omega G(x, t_n) dx \\
&\leq | t_n |^{p^-} K_1 + c_3 \int_\Omega | t_n |^{q_1(x)} - \lambda | t_n |^{p^-} \int_\Omega \frac{G(x, t_n)}{| t_n |^{p^-}} dx \\
&\leq | t_n |^{p^-} K_1 + | t_n |^{q_1^-} K_2 - \lambda | t_n |^{p^-} \int_\Omega \frac{G(x, t_n)}{| t_n |^{p^-}} dx < 0, \quad (3.5)
\end{aligned}$$

and

$$w_n \in B(0,\epsilon) \cap I^-(]-\infty, \rho_2[).$$

Thus,

$$\varphi(u_0) \leq \varphi(w_n) < 0 = \varphi(0), \quad so \quad u_0 \neq 0.$$

`step(4):`

As in step(4) of the theorem 1.1, φ has a mountain pass type critical point u_2. Consequently, u_0, u_1 and u_2 are at least three nontrivial solutions of the problem $(\mathcal{P})$. □

References

1. E. Acerbi, G. Mingione, Regularity results for stationary electrorheological fluids, Arch. Ration. Mech.Anal.164 (2002) 213-259.
2. R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
3. A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and application, J.Funct.Anal.14 (1973)349-381.
4. A. Ayoujil, A.R. El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponent. Nonlinear Anal. T.M.A. (2009)4916-4926.
5. G.Bonanno, B.Di Bella, A boundary value problem for fourth-order elastic beam equations, J.Math.Anal.Appl. 343(2) (2008)1116-1176.
6. S.G. Deng, A local mountain pass theorem and applications to a double perturbed $p(x)$-laplacien equations, J.Applied Mathematics and computation 211 (2009)234-241.
7. X.L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J.Math.Anal.Appl. 263(2001)424-446.
8. X.L. Fan, Shao-Gao Deng, Remarks on Ricceri's variational principle and applications to the $p(x)-$Laplacian equations, J.Nonlinear Anal 67 (2007)3064-3075.
9. A. Ferrero, G. Warnault, On a solutions of second and fourth order elliptic with power type nonlinearities, Nonlinear Anal.T.M.A 70 (2009)2889-2902.
10. C. Li, C-L Tang, Three solutions for a class of quasilinear elliptic systems involving the (p,q)-Laplacian, Nonlinear Anal. 69 (2008)3322-3329.
11. A.M. Micheletti, A. Pistoia, Multiplicity rsults for a fourth-order semilinear elliptic problem, Nonlinear Anal 31(7) (1998)895-908.
12. M. Mihăilescu, V. Rădulescu, A multiplicity result for nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc.R.Soc.Lond.Ser.A 462 (2006) 2625-2641.
13. J. Musielak, Orlicz Spaces and Modular Spaces, in:Lecture Notes in Mathematics,vol.1043, Springer, Berlin, 1983.
14. M. Rủžička, Electrorheological Fluids: Modeling and Mthematical Theory,in:Lecture notes in Math, vol.1748,Springer-Verlag,Berlin,2000.
15. E. Zeidler, Non linear Fonction Analysis and its Applications, vol.II/B:Nonlinear Monotone Operators, Spring, New york,1990.
16. J. Zhang, S. Li, Multiple nontrivial solutions for some fourth order semilinear elliptic problems, Nonlinear Anal(60) (2005)221-230.

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