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On a Class of n-Normed Sequences Related to the ℓ_p-Space

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abstract: In this paper we introduce the class of n-normed sequences related to the p-absolutely summable sequence space. Some properties of this sequence space like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving this sequence space.

Key Words: n-norm, n-Banach space, symmetricity, solidness, convergence free, completeness.

Contents

1. Introduction

The notion of n-normed space was studied at the initial stage by Gahler [8], Gunawan [9], Misiak [10] and many others from different aspects. Diminnie, Gahler and White [2,3] Diminnie and White [4], Gahler [8] and many others have investigated the notion of 2-normed spaces.

Let $n \in N$ and X be a real vector space. A real valued function on X^n satisfying the following four properties:

1. $\|(z_1, z_2, ..z_n)\|_n = 0$ if and only if $z_1, z_2, ..z_n$ are linearly dependent;
2. $\|(z_1, z_2, ..z_n)\|_n$ is invariant under permutation;
3. $\|(z_1, z_2, .., z_{n-1}, \alpha z_n)\|_n = |\alpha| \|z_1, z_2, ..z_n\|_n$, for all $\alpha \in R$;
4. $\|(z_1, z_2, ..z_{n-1}, x + y)\|_n \leq \|(z_1, z_2, ..z_{n-1}, x)\|_n + \|(z_1, z_2, ..z_{n-1}, y)\|_n$;

is called an n-norm on X and the pair $(X, \|, ., .\|_n)$ is called an n-normed space.

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Sargent [12] introduced the sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Esi [5], Et, Altin and Altinok [6], Fistikci, Acikgoz and Esi [7], Rath and Tripathy [11], Tripathy [13], Tripathy and Sen [24], Tripathy and Mahanta [17] and many others. In this article we introduce the n-normed space $(m(\phi), \|...\|_n)$ and investigate its different algebraic and topological properties.

2. Definitions and Background

In the recent past different classes of sequences have been introduced and their different algebraic and topological properties have been investigated by Altin, Et and Tripathy [1], Tripathy [14], Tripathy and Dutta [15,16], Tripathy and Hazarika [17,18], Tripathy and Mahanta [20,21,22], Tripathy and Sarma [23], Tripathy and Sen [25] and many others.

A sequence (x_k) in an n-normed space is said to be convergent to $x \in X$ if,

$$\lim_{k\to\infty} \|(z_1, z_2, ..z_{n-1}, x_k - x)\|_n = 0, \text{for all } z_1, z_2....z_{n-1} \in X.$$

A sequence (x_k) in an n-normed space is called Cauchy (with respect to n-norm) if,

$$\lim_{k,j\to\infty} \|(z_1, z_2, ..z_{n-1}, x_k - x_j)\|_n = 0, \text{for all } z_1, z_2....z_{n-1} \in X.$$

If every Cauchy sequence in X converges to an $x \in X$, then X is said to be complete (with respect to the n-norm). A complete n-normed space is called n-Banach space.

An n-normed sequence space E is said to be solid if $(y_k) \in E$, whenever $(x_k) \in E$ and $\|z_1, z_2, ...z_{n-1}, y_k\|_n \le \|z_1, z_2, ...z_{n-1}, x_k\|_n$ for all $k \in N$.

Let $x = (x_k)$ be a sequence, then $S(x)$ denotes the set of all permutations of the elements of (x_k) i.e. $S(x) = \{(x_{\pi(n)}) : \pi \text{ is a permutation of N}\}$. A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$.

A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ with $y_k = 0$ whenever $x_k = 0$.

A sequence space E is said to be monotone if E contains the canonical preimages of all its step spaces.

Lemma 2.1. *A sequence space E is solid implies that E is monotone.*

Let $\wp_s$ be the class of all subsets of N those do not contain more than s number of elements.

Throughout $\{\phi_n\}$ is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$, for all $n \in N$.

The space $m(\phi)$ introduced by Sargent [12] is defined by,

$$m(\phi) = \left\{ (x_k) \in w : \parallel x \parallel_{m(\phi)} = \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty \right\}.$$

In this article we introduce the following sequence space:

$$(m(\phi), \|...\|_n) = \left\{ (x_k) \in w : \|(x_k)\|_{n,m(\phi)} = \sup_{s\geq 1, \sigma \in \wp} \frac{1}{\phi_s} \sum_{k\in\sigma} \|(z_1, z_2, ...z_{n-1}, x_k)\|_n < \infty \right\}.$$

3. Main Results

Theorem 3.1. *$(m(\phi), \|...\|_n)$ is an n- Banach space with respect to the n-norm defined by,*

$$\|x\|_{n,m(\phi)} = \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \|(z_1, z_2, ..., z_{n-1}, x_k)\|_n$$

for all $z_1, z_2, ..., z_{n-1} \in X$ where X is a Banach space.

Proof: Let $(x^{(i)})$ be a Cauchy sequence in $(m(\phi), \|...\|_n)$ such that, $x^{(i)} = (x_k^{(i)})_{k=1}^{\infty}$.

Then we have for a given $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that,

$$\|(x^{(i)} - x^{(j)})\|_{n,m(\phi)} < \frac{\varepsilon}{\phi_1}, \text{ for all } i, j \geq n_0.$$

Which implies,

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \|(z_1, z_2, ..z_{n-1}, (x_k^{(i)} - x_k^{(j)}))\|_n < \frac{\varepsilon}{\phi_1}, \text{ for all } i, j \geq n_0.$$

On taking $s = 1$ and varying σ over $\wp_s$, we get,

$$\frac{1}{\phi_1} \sum_{k\in\sigma} \|(z_1, z_2, ..z_{n-1}, (x_k^{(i)} - x_k^{(j)}))\|_n < \frac{\varepsilon}{\phi_1}, \text{ for all } i, j \geq n_0.$$

$$\Rightarrow \|(z_1, z_2, ..z_{n-1}, (x_k^{(i)} - x_k^{(j)}))\|_n < \varepsilon, \text{ for all } i, j \geq n_0. \tag{1}$$

i.e.$(x_k^{(i)})$ is a Cauchy sequence in X. Since X is a Banach space, so $(x_k^{(i)})$ converges in X.

Let,

$$\lim_{i\to\infty} x_k^{(i)} = x_k, \text{ for each } k \in N. \tag{2}$$

We have to prove that, $\lim\limits_{i\to\infty} x^{(i)} = x$ and $x \in (m(\phi), \|...\|_n)$.

Since $(x^{(i)})$ is a Cauchy sequence in $(m(\phi), \|...\|_n)$, so we get, for each $k \in N$, there exists a positive integer $n_0(\varepsilon)$ such that,

$$\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \|(z_1, z_2, ..z_{n-1}, (x_k^{(i)} - x_k^{(j)}))\|_n < \varepsilon, \text{ for all } i, j \geq n_0.$$

Taking limit as $j \to \infty$, we have,

$$\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \|(z_1, z_2, ..z_{n-1}, (x_k^{(i)} - x_k))\|_n < \varepsilon, \text{ for all } i \geq n_0.$$

Which implies,

$$\|(x^{(i)} - x)\|_{n,m(\phi)} < \varepsilon, \text{ for all } i \geq n_0.$$

i.e. $\lim\limits_{i\to\infty} x^{(i)} = x$.

Now we have for all $i \geq n_0$,

$$\begin{aligned} \|(x)\|_{n,m(\phi)} &\leq \|(x + x^{(i)} - x^{(i)})\|_{n,m(\phi)} \\ &\leq \|(x - x^{(i)})\|_{n,m(\phi)} + \|(x^{(i)})\|_{n,m(\phi)} \\ &< \varepsilon + M, \text{ for all } i \geq n_0. \end{aligned}$$

i.e.$\|(x^{(i)})\|_{n,m(\phi)} < \infty$.

Hence, $x \in (m(\phi), \|...\|_n)$.

Hence $(m(\phi), \|...\|_n)$ is an n-Banach space. □

Theorem 3.2. *The class of sequences* $(m(\phi), \|..\|_n)$ *is solid.*

Proof: Let $(x_k) \in (m(\phi), \|...\|_n)$ and $(y_k) \in w$ be such that,

$$\|(z_1, z_2, ..., z_{n-1}, y_k)\|_n \leq \|(z_1, z_2, ..., z_{n-1}, x_k)\|_n.$$

Then,

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \|(z_1, z_2, ..z_{n-1}, y_k)\|_n \leq \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \|(z_1, z_2, ..z_{n-1}, x_k)\|_n$$

Thus we have, $\|(y_k)\|_{n,m(\phi)} \leq \|(x_k)\|_{n,m(\phi)} < \infty$.

Which implies $(y_k) \in (m(\phi), \|...\|_n)$.

Hence $(m(\phi), \|...\|_n)$ is solid. □

We have the following result which follows from the Lemma 2.1 and Theorem 3.2.

Corollary 3.3. $(m(\phi), \|..\|_n)$ *is monotone.*

Theorem 3.4. *The class of sequences* $(m(\phi), \|...\|_n)$ *is symmetric.*

Proof: Let $(x_k) \in (m(\phi), \|...\|_n)$ and (y_k) be a rearrangement of (x_k). Then $y_k = x_{m_k}$, for some $k \in N$.

We have, $\|(x_k)\|_{n,m(\phi)} = \|(x_{m_i})\|_{n,m(\phi)} = \|(y_k)\|_{n,m(\phi)}$.

Which implies that, $(y_{m_k}) \in (m(\phi), \|...\|_n)$.

Hence $(m(\phi), \|...\|_n)$ is symmetric. □

Theorem 3.5. *The class of sequences* $(m(\phi), \|...\|_n)$ *is not convergence free.*

Proof: The proof follows from the following example:

Example 3.6.

Let $\phi_s = s$, for all $s \in N$ and $n = 2$.

Let $x_k = k^{-1}$, for all $k \in N$ and $z = 2$.

Then, we have, $\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \|(z, x_k)\|_2$, for all $z \in X$.

Where,

$$\begin{aligned}\|(z,x_k)\|_2 &= \text{area of the rectangle with the vertices } (0,0),(z,0),(0,x_k) \text{ and } (z,x_k)\\ &= x_k \times z\\ &= k^{-1}\times 2,\ \text{for all } k\in N.\end{aligned}$$

Thus we have, $\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{s}\sum_{k\in\sigma}\|(z,x_k)\|_2 < \infty.$

Which implies $(x_k)\in(m(\phi),\|...\|_2)$.

Let us take another sequence (y_k) such that, $y_k = k$, for all $k\in N$.

Then we get,

$$\begin{aligned}\|(z,y_k)\|_2 &= \text{area of the rectangle with the vertices } (0,0),(z,0),(0,y_k) \text{ and } (z,y_k)\\ &= y_k \times z\\ &= k\times 2,\ \text{for all } k\in N.\end{aligned}$$

Thus we have, $\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{s}\sum_{k\in\sigma}\|(z,y_k)\|_2 = \infty.$

i.e. $(y_k)\notin(m(\phi),\|...\|_2)$.

Hence, $(m(\phi),\|...\|_n)$ is not convergence-free.

The proof of the following result is easy, so omitted. □

Proposition 3.7. *(a)* $(\ell,\|...\|_n)\subseteq(m(\phi),\|...\|_n)\subseteq(\ell_\infty,\|...\|_n)$.
(b) $(m(\phi),\|...\|_n)=(\ell,\|...\|_n)$ *if and only if* $\lim_{s\to\infty}\phi_s<\infty$.
(c) $(m(\phi),\|...\|_n)=(\ell_\infty,\|...\|_n)$ *if and only if* $\lim_{s\to\infty}\left(\frac{\phi_s}{s}\right)>0$.

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