On the Gauss Map of Ruled Surfaces of Type II in 3-Dimensional Pseudo-Galilean Space

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Abstract: In this paper, ruled surfaces of type II in a three-dimensional Pseudo-Galilean space are given. By studying its Gauss map and Laplacian operator, we obtain a classification of ruled surfaces of type II in a three-dimensional Pseudo-Galilean space.

Key Words: Pseudo-Galilean Space, Ruled Surfaces, Gauss Map.

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1. Introduction

The classification of submanifold in Euclidean or Non-Euclidean spaces has been of particular interest for Geometers. In 1983, the authors classified spacelike ruled surfaces in a three-dimensional Minkowski space \( \mathbb{R}^3_1 \) [13], and De Woestijne [11] extended it to the Lorentz version in 1998. In late 1970s Chen [4,5] introduced the notion of Euclidean immersions of finite type. In a framework of theory of finite type, Chen and Piccinni [6] characterized the submanifold satisfying the condition \( \Delta G = \lambda G \) \((\lambda \in \mathbb{R})\), where \( \Delta \) in the Laplacian of the induced metric and \( G \) the Gauss map for submanifold. Submanifold of Euclidean and pseudo-Euclidean spaces with finite type Gauss map also studied by following geometers. (cf. [7,9,12,15,16], etc.) On the otherhand, for the Gauss map of a surface in a three-dimensional Euclidean space \( \mathbb{R}^3 \) the following theorem is proved by Boikoussis and Blair [8].

**Theorem 1.1.** The only ruled surfaces in \( \mathbb{R}^3 \) whose Gauss map \( \xi \) satisfies

\[
\Delta \xi = A \xi, \quad A \in \text{Mat}(3, \mathbb{R})
\]

are locally the plane and the circular cylinder.

Also, for the Lorentz version Choi [15] investigated ruled surfaces with non-null base curve satisfying the condition (1.1) in a three-dimensional Minkowski space \( \mathbb{R}^3_1 \).

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It seems to be interesting to investigate the Pseudo-Galilean version of the above theorem. Now, let \( G^3_1 \) be a three-dimensional Pseudo-Galilean space with standard coordinate system \( \{X_A\} \). Let \( S^m_m \) (resp. \( H^m_m \)) be an \( m \)-dimensional de Sitter space (resp. a hyperbolic space) in a \( m+1 \)-dimensional Pseudo-Galilean space \( G^1_{m+1} \). We denote by \( X_m(\varepsilon) \) a de Sitter space \( S^1_m(1) \) or a hyperbolic space \( H^1_m(-1) \), according as \( \varepsilon = 1 \) or \( \varepsilon = -1 \). Let \( X \) be a surface in \( G^3_1 \) and \( \xi \) be a unit vector field normal to \( X \). Then, for any point \( \rho \) in \( X \), we can regard \( \xi(\rho) \) as a point in \( H^2_m(-1) \) or \( S^1_m(1) \) by translating parallelly to the origin in ambient space \( G^3_1 \), according as \( \varepsilon = \pm 1 \). The map \( \xi \) of \( X \) into \( X_{2}(\varepsilon) \) is called a Gauss map of \( X \) in \( G^3_1 \). Then we prove the following

**Theorem 1.2.** The only ruled surfaces of type II in \( G^3_1 \) whose Gauss map \( \xi : X \to X_{2}(\varepsilon) \) satisfies (1.1) are locally the following spaces:

1. \( G^2_1 \) and \( S^1_1 \times \mathbb{R} \) if \( \varepsilon = -1 \)
2. \( G^2_1 \) and \( H_1 \times \mathbb{R} \) if \( \varepsilon = -1 \).

The theorem is proved in section 3.

2. Ruled Surfaces

First of all, we recall fundamental properties in three-dimensional Pseudo-Galilean space.

Differential geometry of the Galilean space \( G^3_3 \) has been largely developed in O. Röschel’s paper \([14]\).

The Pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature \( (0,0,+,-) \), explained in \([10]\). The absolute of the Pseudo-Galilean geometry is an ordered triple \( \{w,f,I\} \) where \( w \) is the ideal (absolute) plane, \( f \) is line in \( w \) and \( I \) is the fixed hyperbolic involution of points of \( f \).

The group of motions of \( G^3_1 \) is a six-parameter group given (in affine coordinates) by

\[
\begin{align*}
\bar{x} &= a + x \\
\bar{y} &= b + cx + ych\varphi + zsh\varphi \\
\bar{z} &= d + ex + ysh\varphi + zch\varphi.
\end{align*}
\]

As in \([1]\), Pseudo-Galilean scalar product \( g \) can be written as

\[
g(v_1, v_2) = \begin{cases} 
  x_1 x_2 & \text{if } x_1 \neq 0 \lor x_2 \neq 0 \\
  y_1 y_2 - z_1 z_2 & \text{if } x_1 = 0 \land x_2 = 0
\end{cases}
\] (2.1)

where \( v_1 = (x_1, y_1, z_1) \), \( v_2 = (x_2, y_2, z_2) \).

It leaves invariant the Pseudo-Galilean norm of the vector \( v = (x, y, z) \) defined by

\[
\|v\| = \begin{cases} 
  x, & x \neq 0 \\
  \sqrt{|y^2 - z^2|}, & x = 0
\end{cases} \quad [2].
\]

(2.2)
A vector \( \mathbf{v} = (x, y, z) \) is said to be non-isotropic if \( x \neq 0 \). All unit non-isotropic vectors are of the form \((1, y, z)\). For isotropic vectors \( x = 0 \) holds. There are four types of isotropic vectors: spacelike \((y^2 - z^2 > 0)\), timelike \((y^2 - z^2 < 0)\) and two types of lightlike \((y = \pm z)\) vectors. A non-lightlike isotropic vector is a unit vector if \( y^2 - z^2 = \pm 1 \) \([3]\).

A trihedron \((T_0; e_1, e_2, e_3)\), with a proper origin \( T_0 = (x_0, y_0, z_0) \), is orthonormal in Pseudo-Galilean sense if the vectors \( e_1, e_2, e_3 \) are of the following form:

\[
e_1 = (1, y_1, z_1), \quad e_2 = (0, y_2, z_2), \quad e_3 = (0, \varepsilon z_2, \varepsilon y_2),
\]

with \( y_2^2 - z_2^2 = \delta \), where \( \varepsilon, \delta \) is \(+1\) or \(-1\) \([3]\).

Such trihedron \((T_0; e_1, e_2, e_3)\) is called positively oriented if for its vectors \( \det(e_1, e_2, e_3) = 1 \) holds, i.e. if \( y_2^2 - z_2^2 = \varepsilon \) \([3]\).

Let \( \mathbf{X} \) be a ruled surface, \( r \geq 1 \), in \( G_1^3 \) given by its parametrization

\[
\mathbf{X}(u, v) = r(u) + v\mathbf{a}(u), \quad v \in \mathbb{R}
\]

(2.3)

where \( r \) is the directrix (parametrized by the Pseudo-Galilean arc length) and \( \mathbf{a} \) is a unit generator vector field. according to the position of the striction curve with respect to the absolute figure, we have three types of skew ruled surfaces in \( G_1^3 \).

Throughout this paper, we assume that all objects are smooth unless otherwise mentioned. Now, we define a ruled surface of type II in \( G_1^3 \).

A ruled surface of type II in \( G_1^3 \) is a surface parametrized by

\[
\begin{align*}
\mathbf{X}(u, v) &= r(u) + v\mathbf{a}(u), \\
y, z, a_2, a_3 &\in C^2, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R} \\
y'z' - z'a' &= 1, \quad y'a'_2 - z'a'_3 = 0.
\end{align*}
\]

(2.4)

It is a surface whose striction curve \( r(u) = (0, y(u), z(u)) \) lies in pseudo-Euclidean plane, \( u \) is the arc-length on the striction curve, and whose generators \( \mathbf{a}(u) = (1, a_2(u), a_3(u)) \) are non-isotropic. In particular, if \( \mathbf{a}(u) \) is constant, then it said to be cylindrical, and if it is not so, then the surface is said to be non-cylindrical. Since our discussion is local, we may assume that we always have \( \mathbf{a}'(u) \neq 0 \) in the non-cylindrical case. That is, the direction of the rulings is always changing. \([3]\)

The natural basis \( \{\mathbf{X}_u, \mathbf{X}_v\} \) along the coordinate curves are given by

\[
\begin{align*}
\mathbf{X}_u &= dx(\frac{\partial}{\partial u}) = r' + v\mathbf{a}' \\
\mathbf{X}_v &= dx(\frac{\partial}{\partial v}) = \mathbf{a}.
\end{align*}
\]
Accordingly we see

\[ g(X_u, X_u) = g(r', r') + 2v g(r', a') + v^2 g(a', a') \]
\[ g(X_u, X_v) = 0 \]
\[ g(X_v, X_v) = g(a, a). \]

For the components \( g_{ij} \) of the Pseudo-Galilean metric \( g \) we denote \( (g_{ij}) \) (resp. \( g \)) the inverse matrix (resp. the determinant) of matrix \( (g_{ij}) \). Then the Laplacian \( \Delta \) is given by

\[ \Delta = -\frac{1}{\sqrt{|g|}} \sum \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}). \quad (2.5) \]

3. Cylindrical Ruled Surfaces

In this section we are concerned with cylindrical ruled surfaces. Let \( X \) be a cylindrical ruled surface swept out by the vector field \( a \) along the base curve \( r \) in \( G_3 \). That is, \( r = r(u) \) is isotropic curve and \( a = a(u) \) is a non-isotropic unit constant vector along orthogonal to \( r \). Then the cylindrical ruled surface \( X \) is only or type II. And \( X \) is parametrized by

\[ X(u, v) = r(u) + va(u), \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R}. \]

Then we get \( g(\xi, \xi) = \varepsilon (= \mp 1) \). Let \( X_2(\varepsilon) \) be a two-dimensional space form as follows:

\[ X_2(\varepsilon) = \begin{cases} S_2(1) \text{ in } G_3^1 \text{ if } \varepsilon = -1 \\ H_2(-1) \text{ in } G_3^1 \text{ if } \varepsilon = 1. \end{cases} \]

Then, for any point \( x \) in \( X \), \( \xi(x) \) can be regarded as a point in \( X_2(\varepsilon) \) and the map \( \xi : X \to X_2(\varepsilon) \) is the Gauss map of \( X \) into \( X_2(\varepsilon) \).

We give here theorem of ruled surface of type II whose Gauss map satisfies

\[ \Delta \xi = A\xi, \quad A \in \text{Mat}(3, \mathbb{R}). \quad (3.1) \]

**Theorem 3.1.** The only cylindrical ruled surface of type II in \( G_3^1 \) whose Gauss map satisfies the condition (3.1) are locally the plane and the hyperbolic cylinder (resp. the Pseudo-Galilean plane and Pseudo-Galilean circular cylinder).

**Proof:** Let \( X \) be a cylindrical ruled surface of type II in \( G_3^1 \) parametrized by

\[ X(u, v) = r(u) + va(u) \]
\[ y, z, a_2, a_3 \in C^2, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R} \]
\[ \left| y'z'' - z'y'' \right| = 1, \quad y'a_2' - z'a_3' = 0 \quad (3.2) \]
where \( \mathbf{a} \) is a unit constant vector along the curve \( r \) orthogonal to it. That is, it satisfies \( g(r', \mathbf{a}) = 0, g(\mathbf{a}, \mathbf{a}) = 1 \). Acting a Pseudo-Galilean transformation, we may assume that \( \mathbf{a} = (1, 0, 0) \) without loss of generality. Then \( r \) may be regarded as the plane curve \( r(u) = (0, y(u), z(u)) \) parametrized by arc length:

\[
g(r', r') = y'^2 - z'^2 = \varepsilon \quad (\varepsilon = \mp 1).
\]

(3.3)

The Gauss map \( \xi \) is given by

\[
\begin{align*}
\xi^1 &= (0, \varepsilon z' \, u, \varepsilon y' \, u) \\
\xi^2 &= (a_{12} \varepsilon z' + a_{13} y', a_{22} \varepsilon z' + a_{23} y', a_{32} \varepsilon z' + a_{33} y').
\end{align*}
\]

where \( A = (a_{ij}) \) is the constant matrix.

Now, in order to prove this theorem we may solve this equation and obtain the solution \( y \) and \( z \). First we consider that \( \varepsilon = -1 \). So we get \( g(r', r') = y'^2 - z'^2 = -1 \). Accordingly we can parametrize as follows:

\[
\begin{align*}
y' &= \sinh \theta, \\
z' &= \cosh \theta
\end{align*}
\]

(3.5)

where \( \theta = \theta(u) \). Differentiating (3.5), we obtain

\[
\begin{align*}
y'' &= \theta' \cosh \theta, \\
y''' &= \theta'' \cosh \theta + (\theta')^2 \sinh \theta \\
z'' &= \theta' \sinh \theta, \\
z''' &= \theta'' \sinh \theta + (\theta')^2 \cosh \theta
\end{align*}
\]

(3.6)

By (3.4), (3.5) and (3.6) we have

\[
\begin{align*}
-\theta'' \sinh \theta - (\theta')^2 \cosh \theta &= a_{22} \cosh \theta + a_{23} \sinh \theta \\
-\theta' \cosh \theta - (\theta')^2 \sinh \theta &= a_{32} \cosh \theta + a_{33} \sinh \theta,
\end{align*}
\]

which give

\[
\begin{align*}
\theta'' &= (a_{22} - a_{33}) \cosh \theta \sinh \theta + a_{23} \sinh^2 \theta - a_{32} \cosh^2 \theta \\
(\theta')^2 &= (a_{32} - a_{23}) \cosh \theta \sinh \theta + a_{33} \sinh^2 \theta - a_{22} \cosh^2 \theta.
\end{align*}
\]

(3.7) (3.8)

Differentiating (3.8), we get

\[
2\theta' \theta''' = \theta' \{(a_{22} - a_{33})(\cosh^2 \theta + \sinh^2 \theta) + 2(a_{33} - a_{22}) \sinh \theta \cosh \theta\}.
\]

Substituting (3.7) into this equation, we get
\[ \theta' \left\{ 4(a_{22} - a_{33}) \cosh \theta \sinh \theta + (a_{23} - 3a_{32}) \cosh^2 \theta + (3a_{23} - a_{32}) \sinh^2 \theta \right\} = 0. \] (3.9)

We suppose that \( \theta' \neq 0 \). By (3.4) and (3.9) we get

\[ a_{22} = a_{33} \quad \text{and} \quad a_{12} = a_{13} = a_{23} = a_{32} = 0, \] (3.10)

because \( \sinh \theta \cosh \theta, \sinh^2 \theta \) and \( \cosh^2 \theta \) are linearly independent functions of \( \theta = \theta(u) \). Combining the above equations with (3.8) gives

\[ \theta = \mp \frac{1}{r} u + c_0, \quad r \neq 0, \quad c_0 \in \mathbb{R} \]

where \( -\frac{1}{r^2} = a_{22} = a_{33} \). Accordingly we have

\[ y = \mp r \sinh \theta + c_1, \quad c_1 \in \mathbb{R} \]
\[ z = \mp r \cosh \theta + c_2, \quad c_2 \in \mathbb{R}. \]

This representation gives us to

\[ (y - c_1)^2 - (z - c_2)^2 = r^2, \quad r \neq 0. \]

We denote by \( S^1_1 \left( r, (c_1, c_2) \right) \) the pseudo-circle centered at \( (c_1, c_2) \) with radius \( r \) in the Pseudo-Galilean plane \( G^2_2 \) (the \( yz \)-plane). By the above equation the curve \( r \) is contained in \( S^1_1 \left( r, (c_1, c_2) \right) \) and hence the ruled surface \( X \) is contained in the Pseudo-Galilean circular cylinder \( S^1_1 \times \mathbb{R} \).

Now, we consider that \( \varepsilon = 1 \). So we get \( g(r', r') = y'^2 - z'^2 = 1 \). Accordingly we can parametrize as follows:

\[ y' = \cosh \theta, \quad z' = \sinh \theta \]

where \( \theta = \theta(u) \). By similar discussion to that of the above \( \varepsilon = -1 \) we can get

\[ \theta' \left\{ 4(a_{22} - a_{33}) \cosh \theta \sinh \theta + (a_{23} - 3a_{32}) \cosh^2 \theta + (a_{23} - 3a_{32}) \sinh^2 \theta \right\} = 0. \] (3.11)

We suppose that \( \theta' \neq 0 \). By (3.4) and (3.11) we get

\[ a_{22} = a_{33} \quad \text{and} \quad a_{12} = a_{13} = a_{23} = a_{32} = 0, \]

which yields that

\[ \theta = \mp \frac{1}{r} u + c_0, \quad r \neq 0, \quad c_0 \in \mathbb{R} \]

where \( \frac{1}{r^2} = a_{22} = a_{33} \). Accordingly we have

\[ y = \mp r \sinh \theta + c_1, \quad c_1 \in \mathbb{R} \]
\[ z = \mp r \cosh \theta + c_2, \quad c_2 \in \mathbb{R}. \]
This representation gives us to
\[(y - c_1)^2 - (z - c_2)^2 = -r^2, \quad r \neq 0.\]

We denote by \(H_1(r, (c_1, c_2))\) the hyperbolic circle centered at \((c_1, c_2)\) with radius \(r\) in the Pseudo-Galilean plane \(G^1_2\) (the \(yz\)-plane). By the above equation the curve \(r\) is contained in \(H_1(r, (c_1, c_2))\) and hence the ruled surface \(X\) is contained in the hyperbolic cylinder \(H_1 \times \mathbb{R}\).

**Example 3.2.** A Pseudo-Galilean circular cylinder
\[S^1_1 \times \mathbb{R} = \{(x, y, z) \in G^3_3 : y^2 - z^2 = r^2, \quad r \neq 0\}\]
is a cylindrical ruled surface of type II with base curve \(r(u) = (0, r \sinh \frac{u}{r}, r \cosh \frac{u}{r})\) and director curve \(a(u) = (1, 0, 0)\). The Gauss map is given by
\[\xi = (0, \cosh \frac{u}{r}, \sinh \frac{u}{r})\]
and the Laplacian \(\Delta \xi\) of the Gauss map \(\xi\) can be expressed as
\[\Delta \xi = -\frac{1}{r^2} \xi.\]

Hence the Pseudo-Galilean circular cylinder satisfies (3.1) with
\[A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & -\frac{1}{r^2} & 0 \\ a_{31} & 0 & -\frac{1}{r^2} \end{bmatrix}.\]

**Example 3.3.** A hyperbolic cylinder
\[H_1 \times \mathbb{R} = \{(x, y, z) \in G^3_3 : y^2 - z^2 = -r^2, \quad r \neq 0\}\]
is a cylindrical ruled surface of type II with base curve \(r(u) = (0, r \sinh \frac{u}{r}, r \cosh \frac{u}{r})\) and director curve \(a(u) = (1, 0, 0)\). The Gauss map is given by
\[\xi = (0, \sinh \frac{u}{r}, \cosh \frac{u}{r})\]
and the Laplacian \(\Delta \xi\) of the Gauss map \(\xi\) can be expressed as
\[\Delta \xi = \frac{1}{r^2} \xi.\]

Hence the hyperbolic cylinder satisfies (3.1) with
\[A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & \frac{1}{r^2} & 0 \\ a_{31} & 0 & \frac{1}{r^2} \end{bmatrix}.\]
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