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On a Class of Generalized Sequences Related to the ℓ^p Space Defined by Orlicz Functions

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ABSTRACT: In this article, we introduce the sequence space $m(M, A, \phi, q)$ on generalizing the sequence space $m(\phi)$ which was defined by Sargent [8], defined by Orlicz functions and infinite matrices. We study its different properties like solidity, completeness, etc. Also we obtain some inclusion results involving the sequence space $m(M, A, \phi, q)$.

Key Words: Seminorm, complete, Orlicz function, solid space.

Contents

1. Introduction

Throughout the article $w(X)$ denotes the space of all sequences with elements in (X, q), where (X, q) denote a seminormed space, seminormed by q. The zero sequence is denoted by $\theta = (0, 0, 0, ...)$, where θ is zero element in (X, q).

The sequence space $m(\phi)$ was introduced by Sargent [8]. He studied some of its properties and obtained its relationship with the space ℓ^p. Later on, it was investigated from sequence space point of view and related with summability theory by Bilgin [1], Esi [2,3], Rath and Tripathy [7], Tripathy [9], Tripathy and Sen [10], and many others.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $T > 0$, such that $M(2u) \le TM(u)$ $(u \ge 0)$.

Remark 1.1. *An Orlicz function satisfies the inequality $M(\lambda u) \le \lambda M(u)$ for all λ with $0 < \lambda < 1$.*

Lindenstrauss and Tzafriri [5] used the idea of Orlicz functions to construct Orlicz sequence space,

$$\ell_M = \left\{ x = (x_k) : \sum_{k \ge 1} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

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The sequence space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k \geq 1} M \left(\frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

becomes a Banach space which is called on Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [6], Esi and Et [4], Tripathy and Mahanta [11,12], Tripathy and Sarma [13,14,15], Tripathy and Borgohain [16], Tripathy and Hazarika [17], Tripathy and Dutta [18], and many others.

Let P_s denotes the class of all subsets on $\mathbb{N}$, the natural numbers, those do not contain more than s elements. Throughout (ϕ_n) represents a non decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.

The sequence space $m(\phi)$ introduced by Sargent [8] is defined as follows:

$$m(\phi) = \left\{ x = (x_k) \in w : \ \|x_k\| = \sup_{s \geq 1,\ \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ for all sequences (α_k) of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_k) \in E$ implies $\left(x_{\pi(k)}\right) \in E$, where $\pi(k)$ is a permutation of the elements of $\mathbb{N}$.

A sequence space E is said to be monotone if E contains the canonical pre images of all its step spaces.

The following result will be used for establishing the result of this article.

Lemma 1.2. *A sequence space E is solid implies E is monotone.*

For a given infinite matrix $A = (a_{ik})_{i,k \geq 1}$ the operators A_i are defined for any integer $i \geq 1$, by

$$A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$$

where $x = (x_k)_{k \geq 1}$, the series intervening on the right hand being convergent. In this article we introduce the following sequence spaces.

$$\ell_\infty(M, A, q) = \left\{ x = (x_k) \in w(X) : \sup_{i \geq 1} M \left(q \left(\frac{A_i(x)}{\rho} \right) \right) < \infty, \text{ for some } \rho > 0 \right\},$$

$$\ell_\infty(M, A, q) = \left\{ x = (x_k) \in w(X) : \sum_{i=1}^{\infty} M \left(q \left(\frac{A_i(x)}{\rho} \right) \right) < \infty, \text{ for some } \rho > 0 \right\}$$

and

$$m(M, A, \phi, q) = \left\{ x = (x_k) \in w(X) : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{i \in \sigma} M \left(q \left(\frac{A_i(x)}{\rho} \right) \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

2. Main Results

In this section we prove some results involving the sequence spaces $m(M, A, \phi, q)$, $\ell_\infty(M, A, q)$ and $\ell_1(M, A, q)$.

Theorem 2.1. *$m(M, A, \phi, q)$, $\ell_\infty(M, A, q)$ and $\ell_1(M, A, q)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof: Let $(x_k), (y_k) \in m(M, A, \phi, q)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 suh that

$$\sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right) < \infty$$

and

$$\sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right) < \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing convex function and q is a seminorm, we have

$$\begin{aligned}\sum_{i\in\sigma} M\left(q\left(\frac{A_i(\alpha x+\beta y)}{\rho_3}\right)\right) &\leq \sum_{i\in\sigma} M\left(q\left(\frac{A_i(\alpha x)}{\rho_3}\right)+q\left(\frac{A_i(\beta y)}{\rho_3}\right)\right)\\ &\leq \sum_{i\in\sigma} M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right)+\sum_{i\in\sigma} M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right).\end{aligned}$$

So,

$$\begin{aligned}&\sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i(\alpha x+\beta y)}{\rho_3}\right)\right)\\ \leq\ &\sup_{s\geq 1,\ \sigma\in P} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right)+\sup_{s\geq 1,\ \sigma\in P} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right).\end{aligned}$$

Thus, $(\alpha x_k+\beta y_k) \in m(M, A, \phi, q)$. Hence $m(M, A, \phi, q)$ is a linear space.

The proof for the cases $\ell_\infty(M, A, q)$ and $\ell_1(M, A, q)$ are routine work in view of the above proof. □

Theorem 2.2. *$\ell_1(M, A, q) \subset m(M, A, \phi, q) \subset \ell_\infty(M, A, q)$.*

Proof: Let $(x_k) \in \ell_1(M, A, q)$. Then, for some $\rho > 0$, we have

$$\sum_{i=1}^{\infty} M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) < \infty.$$

Since (ϕ_n) is monotonic increasing, so we have

$$\frac{1}{\phi_s}\sum_{i\in\sigma} M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) \leq \frac{1}{\phi_1}\sum_{i\in\sigma} M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) \leq \frac{1}{\phi_1}\sum_{i=1}^{\infty} M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) < \infty.$$

Hence,

$$\sup_{s\geq 1,\ \sigma\in P}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)<\infty.$$

Thus $(x_k)\in m(M,A,\phi,q)$. Therefore $\ell_1(M,A,q)\subset m(M,A,\phi,q)$. Next, let $(x_k)\in m(M,A,\phi,q)$. Then, for some $\rho>0$, we have

$$\sup_{s\geq 1,\ \sigma\in P}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)<\infty.$$

Hence,

$$\sup_{s\geq 1,\ \sigma\in P}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)<\infty \text{ (on taking cardinality of } \sigma \text{ to be 1).}$$

Thus, $(x_k)\in\ell_\infty(M,A,q)$. Therefore $m(M,A,\phi,q)\subset\ell_\infty(M,A,q)$. This completes the proof. □

Theorem 2.3. *The space $m(M,A,\phi,q)$ is a seminormed space seminormed by*

$$h(x)=\inf\left\{\rho>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)\leq 1\right\}.$$

Proof: Clearly $h((x_k))\geq 0$ for all $(x_k)\in m(M,A,\phi,q)$ and $h(\theta)=0$. Let (x_k), $(y_k)\in m(M,A,\phi,q)$. Then there exist $\rho_1>0$ and $\rho_2>0$ be such that

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right)\leq 1$$

and

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right)\leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then we have

$$\begin{aligned}
\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x+y)}{\rho}\right)\right) &\leq \sup_{s\geq 1,\ \sigma\in P_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x+y)}{\rho_1+\rho_2}\right)\right)\\
&\leq \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}\left\{\frac{\rho_1}{\rho_1+\rho_2}M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right)+\frac{\rho_2}{\rho_1+\rho_2}M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right)\right\}\\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right)\\
&\quad+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right)\\
&\leq 1.
\end{aligned}$$

Since the ρ's are nonnegative, so we have

$$\begin{aligned} h(x+y) &= \inf\left\{\rho>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x+y)}{\rho}\right)\right)\leq 1\right\} \\ &\leq \inf\left\{\rho>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho_1}\right)\right)\leq 1\right\} \\ &\quad +\inf\left\{\rho>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(y)}{\rho_2}\right)\right)\leq 1\right\}. \end{aligned}$$

So, $h(x+y)\leq h(x)+h(y)$. Next for $\lambda\in\mathbb{C}$, without loss of generality, $\lambda\neq 0$, then

$$\begin{aligned} h(\lambda x) &= \inf\left\{\rho>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(\lambda x)}{\rho}\right)\right)\leq 1\right\} \\ &= \inf\left\{|\lambda| r>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{r}\right)\right)\leq 1\right\},\ \text{where } r=\frac{\rho}{|\lambda|}. \end{aligned}$$

Thus,

$$h(\lambda x)=|\lambda|\inf\left\{r>0: \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{r}\right)\right)\leq 1\right\}=|\lambda|\,h(x).$$

This completes the proof of theorem. □

The proof of the following results are consequence of the above theorem.

Proposition 2.4. ***(a)*** *The space $\ell_\infty(M,A,q)$ is a seminormed space, seminormed by*

$$k(x)=\inf\left\{\rho>0: \sup_{i\geq 1,}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)\leq 1\right\}.$$

(b) *The space $\ell_1(M,A,q)$ is a seminormed space, seminormed by*

$$m(x)=\inf\left\{\rho>0: \sum_{i=1}^{\infty}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)\leq 1\right\}.$$

Theorem 2.5. *$m(M,A,\phi,q)\subset m(M,A,\psi,q)$ if and only if $\sup_{s\geq 1}\frac{\phi_s}{\psi_s}<\infty$.*

Proof: Let $\sup_{s\geq 1}\frac{\phi_s}{\psi_s}<\infty$ and $(x_k)\in m(M,A,\phi,q)$. Then, for some $\rho>0$

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)<\infty.$$

So,

$$\begin{aligned}\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\psi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) &\leq \left(\sup_{s\geq 1}\frac{\phi_s}{\psi_s}\right)\\ \sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) &< \infty.\end{aligned}$$

Therefore $(x_k)\in m(M,A,\psi,q)$. Hence we have $m(M,A,\phi,q)\subset m(M,A,\psi,q)$. Conversely, let $m(M,A,\phi,q)\subset m(M,A,\psi,q)$. Suppose that $\sup_{s\geq 1}\frac{\phi_s}{\psi_s}=\infty$. Then there exists a sequence of natural numbers (s_j) such that $\lim_{j\rightarrow}\frac{\phi_{s_j}}{\psi_{s_j}}=\infty$.

Let $(x_k)\in m(M,A,\phi,q)$. Then, for some $\rho>0$,

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)<\infty.$$

Now we have

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\psi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)\geq\left(\sup_{j\geq 1}\frac{\phi_{s_j}}{\psi_{s_j}}\right)\sup_{j\geq 1,\ \sigma\in P_{s_j}}\frac{1}{\phi_{s_j}}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right)=\infty.$$

Therefore $(x_k)\notin m(M,A,\psi,q)$. This is a constradiction. Hence $\sup_{s\geq 1}\frac{\phi_s}{\psi_s}<\infty$. $\square$

The following result is a consequence of Theorem 2.5.

Corollary 2.6. $m(M,A,\phi,q)=m(M,A,\psi,q)$ *if and only if* $\sup_{s\geq 1}\frac{\phi_s}{\psi_s}<\infty$ *and* $\sup_{s\geq 1}\frac{\psi_s}{\phi_s}<\infty$.

Theorem 2.7. *Let M_1 and M_2 be Orlicz functions satisfying Δ_2-condition. Then*

$$m(M_2,A,\phi,q)\subset m(M_1\circ M_2,A,\phi,q).$$

Proof: Let $(x_k)\in m(M_2,A,\phi,q)$. Then there exists $\rho>0$ such that

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M_2\left(q\left(\frac{A_i(x)}{\rho}\right)\right)<\infty.$$

Let $0<\varepsilon<1$ and δ with $0<\delta<1$ such that $M_1(t)<\varepsilon$ for $0\leq t<\delta$.

Let $y_1=M_2\left(q\left(\frac{A_i(x)}{\rho}\right)\right)$ and for any $\sigma\in P_s$, let

$$\sum_{i\in\sigma}M_1(y_i)=\sum_1 M_1(y_i)+\sum_2 M_1(y_i),$$

where the first summation is over $y_i\leq\delta$ and the second is over $y_i>\delta$. By the Remark 1.1, we have

$$\sum_1 M_1(y_i)\leq M_1(1)\sum_1 y_i\leq M_1(2)\sum_1 y_i \tag{1}$$

For $y_i > \delta$, we can write $y_i < \frac{y_i}{\delta} \leq 1 + \frac{y_i}{\delta}$. Since M_1 is non-decreasing and convex, so

$$M_1(y_i) < M_1\left(1 + \frac{y_i}{\delta}\right) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1\left(2\frac{y_i}{\delta}\right).$$

Since M_1 satisfies Δ_2-condition, so

$$M_1(y_i) < \frac{1}{2}T\frac{y_i}{\delta}M_1(2) + \frac{1}{2}T\frac{y_i}{\delta}M_1(2) = T\frac{y_i}{\delta}M_1(2).$$

Hence,

$$\sum_2 M_1(y_i) \leq \max\left(1, T\delta^{-1}M_1(2)\right)\sum_1 y_i \tag{2}$$

By (1) and (2), we have $(x_k) \in m(M_1 \circ M_2, A, \phi, q)$. Therefore, $m(M_2, A, \phi, q) \subset m(M_1 \circ M_2, A, \phi, q)$. This completes the proof. □

Corollary 2.8. *Let M be an Orlicz function satisfying Δ_2-condition. Then,*

(**a**) $m(A, \phi, q) \subset m(M, A, \phi, q)$.
(**b**) $m(A, \phi, q) \subset m(M, A, \psi, q)$ if and only if $\sup_{s\geq 1}\frac{\phi_s}{\psi_s} < \infty$.

Proof: (**a**) Taking $M_2(x) = x$ and $M_1(x) = M(x)$ in the Theorem 2.7, we have the result.

(**b**) From the Theorem 2.5 and (a), we have the result. □

Theorem 2.9. *(**a**) $m(M, A, \phi, q) = \ell_1(M, A, q)$ if and only if $\sup_{s\geq 1}\phi_s < \infty$.*

(**b**) $m(M, A, \phi, q) = \ell_\infty(M, A, q)$ if and only if $\sup_{s\geq 1}\frac{s}{\phi_s} < \infty$.

Proof: (**a**) It is clear that $m(M, A, \psi, q) = \ell_1(M, A, q)$ when $\psi_s = 1$ for all $s \in \mathbb{N}$. By Theorem 2.5, $m(M, A, \phi, q) \subset m(M, A, \psi, q)$ if and only if $\sup_{s\geq 1}\frac{\phi_s}{\psi_s} < \infty$, i.e. $\sup_{s\geq 1}\phi_s < \infty$. Therefore by Theorem 2.2. if and only if $\sup_{s\geq 1}\phi_s < \infty$.

(**b**) We have $m(M, A, \psi, q) = \ell_\infty(M, A, q)$ if $\psi_s = 1$ for all $s \in \mathbb{N}$. By Theorem 2.5 and Theorem 2.2 it follows that $m(M, A, \phi, q) = \ell_\infty(M, A, q)$ if and only if $\sup_{s\geq 1}\frac{s}{\phi_s} < \infty$. This completes the proof. □

Theorem 2.10. *The space $m(M, A, \phi, q)$ is solid and symmetric.*

Proof: Let $(x_k) \in m(M, A, \phi, q)$. Then, for some $\rho > 0$

$$\sup_{s\geq 1,\ \sigma\in P_s}\frac{1}{\phi_s}\sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) < \infty. \tag{3}$$

Let (λ_k) be a sequence of scalars with $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$. Then the solidity of the space follows from (3), Remark 1.1 and the following inequality

$$\sum_{i\in\sigma}M\left(q\left(\frac{A_i(\lambda x)}{\rho}\right)\right) \leq \sum_{i\in\sigma}|\lambda_i|\,M\left(q\left(\frac{A_i(x)}{\rho}\right)\right) \leq \sum_{i\in\sigma}M\left(q\left(\frac{A_i(x)}{\rho}\right)\right).$$

The symmetric of the space follows from the definition of the space and symmetric sequence space. This completes the proof. □

The following result follows from Theorem 2.10 and the Lemma 1.2.

Corollary 2.11. *The space $m(M, A, \phi, q)$ is monotone.*

The proof of the following result is a routine work.

Proposition 2.12. *The spaces $\ell_1(M, A, q)$ and $\ell_\infty(M, A, q)$ are solid and as such are monotone*

The proof of the following result is a routine work.

Proposition 2.13. *Let q_1 and q_2 be seminorms. Then,*

(**a**) $m(M, A, \phi, q_1) \cap m(M, A, \phi, q_2) \subset m(M, A, \phi, q_1 + q_2)$.
(**b**) If q_1 is stronger than q_2, then $m(M, A, \phi, q_1) \subset m(M, A, \phi, q_2)$.
(**c**) $\ell_\infty(M, A, q_1) \cap \ell_\infty(M, A, q_2) \subset \ell_\infty(M, A, q_1 + q_2)$.
(**d**) If q_1 is stronger than q_2, then $\ell_\infty(M, A, q_1) \subset \ell_\infty(M, A, q_2)$.
(**e**) $\ell_1(M, A, q_1) \cap \ell_1(M, A, q_2) \subset \ell_1(M, A, q_1 + q_2)$.
(**f**) If q_1 is stronger than q_2, then $\ell_1(M, A, q_1) \subset \ell_1(M, A, q_2)$.

Theorem 2.14. *Let (X, q) be the complete, then the space $m(M, A, \phi, q)$ is also complete.*

Proof: Let $(x^t)_{t\geq 1}$ be a Cauchy sequence in $m(M, A, \phi, q)$, where $x^t = (x_k^t) \in m(M, A, \phi, q)$ for each $t \in \mathbb{N}$. Let $r > 0$ and $x_0 > 0$ be a fixed. Then for each $\frac{\varepsilon}{rx_0} > 0$, there exists a positive integer t_0 such that

$$h\left(x^t - x^u\right) = \inf\left\{\rho > 0 : \sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M\left(q\left(\frac{A_i\left(x^t - x^u\right)}{\rho}\right)\right) \leq 1\right\} < \frac{\varepsilon}{rx_0}, \tag{4}$$

for all $t, u \geq t_0$. So, we have for all t, $u \geq t_0$, by (4)

$$\sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M\left(q\left(\frac{A_i\left(x^t - x^u\right)}{h\left(x^t - x^u\right)}\right)\right) \leq 1$$

$$\begin{aligned} &\Rightarrow \quad \frac{1}{\phi_1} M\left(q\left(\frac{A_i\left(x^t - x^u\right)}{h\left(x^t - x^u\right)}\right)\right) \leq 1 \\ &\Rightarrow \quad M\left(q\left(\frac{A_i\left(x^t - x^u\right)}{h\left(x^t - x^u\right)}\right)\right) \leq \phi_1, \text{ for all } t,\ u \geq t_0. \end{aligned}$$

We can find $r > 0$ such that $\frac{rx_0}{2}\eta\left(\frac{x_0}{2}\right) > \phi_1$, where η is the kernel associated with Orlicz function M, such that

$$M\left(q\left(\frac{A_i\left(x^t - x^u\right)}{h\left(x^t - x^u\right)}\right)\right) \leq \frac{rx_0}{2}\eta\left(\frac{x_0}{2}\right)$$

$$\Rightarrow q\left(A_i\left(x^t - x^u\right)\right) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}.$$

Hence $A_i\left(x^t\right)_{t\geq 1}$ is a Cauchy sequence in (X, q), which is complete. Therefore for each $k \in \mathbb{N}$, there exists $x_k \in X$ and $x = (x_k)$ such that $q\left(A_i\left(x^t - x\right)\right) \to 0$, as $t \to \infty$. Using the continuity of M and q is seminorm, so for some $\rho > 0$, we have

$$\sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{\lim_{u\to\infty} A_i\left(x^t - x^u\right)}{\rho}\right)\right) \leq 1$$

$$\Rightarrow \sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i\left(x^t - x\right)}{\rho}\right)\right) \leq 1.$$

Now. taking the infimum of such ρ's, by (4) we get

$$\inf\left\{\rho > 0 : \sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(q\left(\frac{A_i\left(x^t - x\right)}{\rho}\right)\right) \leq 1\right\} < \varepsilon, \text{ for all } t \geq t_0.$$

Since $m(M, A, \phi, q)$ is a linear space and $\left(x^t\right)_{t\geq 1}$ and $\left(x - x^t\right)$ are in $m(M, A, \phi, q)$, so it follows that $x = x^t + \left(x - x^t\right) \in m(M, A, \phi, q)$. Hence $m(M, A, \phi, q)$ is complete. This completes the proof. □

3. Conclusion

If one considers a normed linear space $(X, \|.\|)$ instead of seminormed space (X, q), then one will get $m(M, A, \phi, \|.\|)$, which will be normed linear space, normed by

$$\|x\| = \inf\left\{\rho > 0 : \sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\left\|\left(\frac{A_i(x)}{\rho}\right)\right\|\right) \leq 1\right\}.$$

The space $m(M, A, \phi, \|.\|)$ will be a Banach space if $(X, \|.\|)$ is a Banach space. The most of the results proved in the previous section will be true for this space too. Also, giving particular values the matrix A, we obtain some sequences spaces which were defined earlier some authors. For instance, if we take $A = I$ (identity matrix), we obtain the space $m(M, \phi, q)$ which was defined and studied by Tripathy and Mahanta [11].

References

1. T. Bilgin, The sequence space $\ell(p, f, q, s)$ on seminormed spaces, Bull. Calcutta Math. Soc., **86(**1994**),** 295-304.

2. A. Esi, On a class of new type difference sequence spaces related to the space ℓ^p, Far East J. Math. Sci. (FJMS) **13**(2)(2004), 167-172.

3. A. Esi, On a class of sequence related to the ℓ_p space defined by a sequence of Orlicz functions, Mathematicki Vesnik, **62**(1)(2010), 85-93.

4. A. Esi and M. Et, Some new sequence spaces defined by a sequence of Orlicz functions, Indian J. Pure Appl. Math., **31**(8)(2000), 967-972.

5. J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., **10**(1971), 379-390.

6. S. D. Parashar and B. Choudhary, Sequence spaces defined by Orlicz function, Indian J. Pure Appl. Math., **25**(1994), 419-428.

7. D. Rath and B. C. Tripathy, Characterization of Certain matrix operations, J. Orissa Math. Soc., **8**(1989), 121-134.

8. W. L. C. Sargent, Some sequence spaces related to ℓ^p spaces, J. London Math. Soc., **35**(1960), 161-171.

9. B. C. Tripathy, Matrix maps on the power series convergent on the unit disc, J. Analysis **6**(1998), 27-31.

10. B. C. Tripathy and M. Sen, On a new class of sequences related to the space ℓ^p, Tamkang J. Math., **33**(2)(2002), 171-176.

11. B.C. Tripathy and S. Mahanta, On a class of sequences related to the ℓ^p space defined by Orlicz functions, Soochow Journal of Mathematics, **29**(4)(2003), 379-391.

12. B.C. Tripathy and S. Mahanta,On a class of generalized lacunary difference sequence spaces defined by Orlicz function, Acta Mathematica Applicata Sinica (Eng.Ser.) 20(2)(2004), 231-238.

13. B. C. Tripathy and B. Sarma, Sequence spaces of fuzzy real numbers defined by Orlicz functions, Mathematica Slovaca, 58(5)(2008), 621-628.

14. B. C. Tripathy and B. Sarma, Vector valued double sequence spaces defined by Orlicz function, Mathematica Slovaca 59(6)(2009), 767-776.

15. B. C. Tripathy and B. Sarma, Double sequence spaces of fuzzy numbers defined by Orlicz function, Acta Mathematica Scientia, 31**B(**1)(2011), 134-140.

16. B. C. Tripathy and S. Borgogain, The sequence space $m\left(M,\phi,\Delta_m^n,p\right)^F$, Mathematical Modelling and Analysis, 13(4)(2008), 577-586.

17. B. C. Tripathy and B. Hazarika, I-convergent sequence spaces defined by Orlicz function, Acta Mathematica Applicatae Sinica, 27(1)(2011), 149-154.

18. B. C. Tripathy and H. Dutta, On some new paranormed difference sequence spaces defined by Orlicz functions, Kyungpook Mathematical Journal, 50(2010), 59-69.

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