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Spectrum of the Operator $B(f,g)$ on the Vector Valued Sequence Space $c_0(X)$

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ABSTRACT: In this paper we have investigated the spectrum of the operator $B(f,g)$ on the vector valued sequence space $c_0(X)$.

Key Words: Spectrum, resolvent set, matrix map.

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1. Introduction

Spectral theory is an important branch of mathematics due to its application in other branches of science. It has been proved to be a standard tool of mathematical sciences because of its usefulness and application oriented scope in different fields. In numerical analysis, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system. In quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star. In structural mechanics, it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In probability theory, they may determine the rate of convergence of a Markov process.

Summability theory has potential applications in many branches of science. In summability theory, different classes of matrices have been investigated. Characterization of matrix classes are found in Tripathy and Sen [17], Tripathy [10] and many others. There are particular types of summability methods like Nörlund mean, Riesz mean, Euler mean, Abel transformation etc. Matrix methods have been studied from different aspects recently by Altin, Et and Tripathy [4], Tripathy and Baruah [12] and others. Still there is a lot to be explored on spectra

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of some matrix operators transforming one class of sequences into another class of sequences. The spectra of the difference operator has also been investigated on some classes of sequences. Altay and Basar [1,2,3] studied the spectra of difference operator $\triangle$ and generalized difference operator on c_0, c and ℓ_p. Throughout N denote the set of non-negative integers. Okutoyi [8] has investigated the spectra of the Cesàro operator on the sequence space bv_0.

2. Preliminaries and Definition

Sequence spaces has been investigated from different aspects. Sequences of fuzzy numbers have been investigated by Tripathy and Baruah [11], Tripathy and Borgohain [13], Tripathy and Sarma [16] and many others. Recently some vector valued sequence spaces have been investigated by Tripathy and Mahanta [14], Tripathy and Sarma [15] and many others. In his monograph Maddox [7] has written about works done on vector valued sequence spaces as well as matrix transformations involving vector valued sequence spaces. The duals of some sequence spaces is found in Chandra and Tripathy [5].

Let X be a linear space. By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, where X is a Banach space, then the adjoint operator T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$.

Let $T : D(T) \to X$ be a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T \in B(X)$ we associate a complex number α with the operator $(T - \alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T - \alpha I)^{-1}$, denoted by T_α^{-1} is known as the resolvent operator of T.

A regular value is a complex number α of T such that

(R_1) T_α^{-1} exists,

(R_2) T_α^{-1} is bounded and

(R_3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set of T is the set of all such regular values α of T, denoted by $\rho(T)$. Its complement is given by $C \setminus \rho(T)$ in the complex plane C is called the spectrum of T, denoted by $\sigma(T)$. Thus the spectrum $\sigma(T)$ consist of those values of $\alpha \in C$, for which T_α is not invertible.

Throughout $w(X)$ denote the class of all X-valued sequence spaces.

Let $A = (f_{nk})$ be an infinite matrix of continuous linear transformations $f_{nk} : X \to Y$, for all $n, k \in N$ and write $(Ax)_n = \sum\limits_{k=0}^{\infty} f_{nk}(x_k)$, for all $n \in N, x \in D_{00}(A)$, where $D_{00}(A)$ denotes the subspace of $w(X)$ consisting of $x \in w(X)$ for which the sum exists.

For a Banach space $(X, ||.||)$, we consider the following X-valued spaces

$$c_0(X) = \{(x_n) \in w(X) : \{||x_n||\}_{n=1}^{\infty} \in c_0\},$$

$$\ell_\infty(X) = \{(x_n) \in w(X) : \{||x_n||\}_{n=1}^{\infty} \in \ell_\infty\}.$$

These sequence spaces are Banach spaces with respect to norm defined by

$$||x|| = \sup_n ||x_n||.$$

We consider the matrix $A = B(f, g)$ defined by

$$B(f,g) = \begin{pmatrix} f & \bar{0} & \bar{0} & \bar{0} & ... \\ g & f & \bar{0} & \bar{0} & ... \\ \bar{0} & g & f & \bar{0} & ... \\ . & . & . & . & ..., \end{pmatrix}$$

where f, g are bounded linear operators and $\bar{0}$ is the null operator. We shall determine its spectrum on $c_0(X)$

For the following results one may refer to Maddox [7].

Lemma 2.1. *If $(X, ||.||)$ and $(Y, ||.||)$ be two normed linear spaces and $A = (f_{nk})$ be a matrix of bounded linear mappings from X into Y such that*

(a) $\lim_{n\to\infty} f_{nk} = \bar{0}$, for each $n \in N$,

(b) $M = \sup_n \sum_{n=1}^{\infty} ||f_{nk}|| < \infty$.

The convergence in(a) being with respect to the norm in $B(X, Y)$, the space of bounded linear mappings from X to Y, Then A is a bounded linear transformation from $c_0(X)$ into $c_0(Y)$ with $A(c_0(X)) \subset c_0(Y)$ and $||A|| \le M$.

Through out our discussion we consider $f_{nk} : X \to X$ for all $n, k \in N$. We have the following results in view of lemma 2.1.

Corollary 2.2. *$B(f, g) : c_0(X) \to c_0(X)$ is a bounded linear operator and $||B(f, g)|| \le ||f|| + ||g||$.*

Lemma 2.3. *In addition to the conditions of Lemma 1.1, if for each positive integer p there exists a sequence (x_n) in X, with $||x_k|| = 1$ for $1 \le k \le p$ and $x_k = \theta$ for $k > p$ and*

$$||f_{m1}(x_1) + f_{m2}(x_2) + - - - + f_{mp}(x_p)|| = \sum_{k=1}^{p} ||f_{nk}||$$

then $||A|| = M$.

3. Main Result

In this section we determine the spectrum of the operator $B(f,g)$ on the sequence space $c_0(X)$.

Theorem 3.1. *Let $f \in B(X)$ be such that $(f-\alpha I)^{-1}$ exists. Then*

$$\sigma(B(f,g), c_0(X)) = \{\alpha \in C : f = \alpha I \text{ or } ||f-\alpha I|| \leq ||g||\}$$

Proof: First we find $\rho(B(f,g), c_0(X))$. Then the complement of this is our required spectrum.
For each complex number α, consider the operator

$$B(f,g) - \alpha I_* = \begin{pmatrix} f-\alpha I & \bar{0} & \bar{0} & \bar{0} & ... \\ g & f-\alpha I & \bar{0} & \bar{0} & ... \\ \bar{0} & g & f-\alpha I & \bar{0} & ... \\ . & . & . & . & ... \end{pmatrix},$$

where

$$I_* = \begin{pmatrix} I & \bar{0} & \bar{0} & \bar{0} & ... \\ \bar{0} & I & \bar{0} & \bar{0} & ... \\ \bar{0} & \bar{0} & I & \bar{0} & ... \\ . & . & . & . & ... \end{pmatrix},$$

and I is the identity operator defined on X.
It can be easily verified that for each complex number α, $(B(f,g) - \alpha I_*) \in B(c_0(X))$

Let

$$(B(f,g) - \alpha I_*)^{-1} = \begin{pmatrix} p_1 & \bar{0} & \bar{0} & \bar{0} & ... \\ p_2 & p_1 & \bar{0} & \bar{0} & ... \\ p_3 & p_2 & p_1 & \bar{0} & ... \\ . & . & . & . & ... \end{pmatrix}.$$

Then we have

$$\begin{aligned} (f-\alpha I)p_1 &= I \\ gp_1 + (f-\alpha I)p_2 &= \bar{0} \\ gp_2 + (f-\alpha I)p_3 &= \bar{0} \end{aligned}$$

$$- - - - - - - - - - - - - - - - - - \qquad (1)$$

We have $(f-\alpha I)$ is invertible implies $(f-\alpha I)^n$ is invertible for all $n \in N$.

Since $f \neq \alpha I$, thus from (1) we have

$$\begin{aligned} p_1 &= \frac{1}{(f-\alpha I)} \\ p_2 &= \frac{-g}{(f-\alpha I)^2} \\ p_3 &= \frac{g^2}{(f-\alpha I)^3} \end{aligned}$$

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Hence we get,

$$(B(f,g)-\alpha I_*)^{-1} = \begin{pmatrix} \frac{1}{(f-\alpha I)} & \bar{0} & \bar{0} & \bar{0} & ... \\ \frac{-g}{(f-\alpha I)^2} & \frac{1}{(f-\alpha I)} & \bar{0} & \bar{0} & ... \\ \frac{g^2}{(f-\alpha I)^3} & \frac{-g}{(f-\alpha I)^2} & \frac{1}{(f-\alpha I)} & \bar{0} & ... \\ . & . & . & . & ... \end{pmatrix},$$

We claim that $(B(f,g)-\alpha I_*)^{-1} \in B(c_0(X))$, when $||f-\alpha I|| > ||g||$.

Now, $\left\| \frac{(-1)^n g^n}{(f-\alpha I)^{n+1}} \right\| = \frac{||g||^n}{||f-\alpha I||^{n+1}} = \frac{1}{||f-\alpha I||}(\frac{||g||}{||f-\alpha I||})^n \to 0$, if $||f-\alpha I|| > ||g||$.

Thus, $\left\{ \frac{(-1)^n g^n}{(f-\alpha I)^{n+1}} \right\}_{n=0}^{\infty} \to \bar{0}$, with respect to the norm in $B(c_0(X))$, if

$$||f-\alpha I|| > ||g||. \tag{2}$$

Again we have

$$\sup_n \sum_{k=0}^{n} \left\| \frac{(-g)^{n-k}}{(f-\alpha I)^{n-k+1}} \right\| = \frac{1}{||f-\alpha I||} \sup_n \sum_{k=0}^{n} \left\| \frac{g}{f-\alpha I} \right\|^k = \frac{1}{||f-\alpha I||} \sup_n \sum_{k=0}^{n} (\frac{||g||}{||f-\alpha I||})^k < \infty ,$$

when

$$||f-\alpha I|| > ||g||. \tag{3}$$

If $||f-\alpha I|| = ||g||$, then from (3) we have

$$\sup_n \sum_{k=0}^{n} \left\| \frac{(-g)^{n-k}}{(f-\alpha I)^{n-k+1}} \right\| = \infty \tag{4}$$

From (2),(3) and (4)we conclude that

$(B(f,g) - \alpha I_*)^{-1} \in B(c_0(X)$, if $f \neq \alpha I$ and $||f - \alpha I|| > ||g||$.

Therefore, $\rho(B(f,g), c_0(X)) = \{\alpha \in C : f \neq \alpha I$ and $||f - \alpha I|| > ||g||\}$

Hence

$$\sigma(B(f,g), c_0(X)) = C - \rho(B(f,g), c_0(X)) = \{\alpha \in C : f = \alpha I \text{ or } ||f - \alpha I|| \leq ||g||\}$$

This completes the proof. □

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