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# Parametric Equations of General Helices in the Sol Space $\mathfrak{Sol}^3$

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ABSTRACT: In this paper, we study general helices in the $\mathfrak{Sol}^3$. We characterize the general helices in terms of their curvature and torsion. Finally, we find out their explicit parametric equations in the $\mathfrak{Sol}^3$

Key Words: General helix, Sol Space, Curvature, Torsion.

## Contents



## 1. Introduction

A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [9,12] for details) is: A necessary and sufficient condition that a curve be a helix is that the ratio of curvature to torsion be constant.

Helices arise in nanosprings, carbon nanotubes, $\alpha$-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiester bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells.

In this article, we study general helices in the $\mathfrak{Sol}^3$. We characterize the general helices in terms of their curvature and torsion. Finally, we find out their explicit parametric equations in the $\mathfrak{Sol}^3$.

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## 2. Riemannian Structure of Sol Space $\mathfrak{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{2.1}$$

where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i, \tag{2.2}$$

where

$$\boldsymbol{\omega}^1 = e^z dx, \quad \boldsymbol{\omega}^2 = e^{-z} dy, \quad \boldsymbol{\omega}^3 = dz, \tag{2.3}$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.4}$$

**Proposition 2.1.** *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\mathfrak{Sol}^3}$ defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

*where the $(i, j)$-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis*

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\mathfrak{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z), \\ (x, y, z) &\to (x, y + c, z), \\ (x, y, z) &\to \left(e^{-c}x, e^{c}y, z + c\right). \end{aligned}$$

## 3. General Helices in Sol Space $\mathfrak{Sol}^3$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$\begin{aligned} g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{T}) &= 1,\ g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{N})=1,\ g_{\mathfrak{Sol}^3}(\mathbf{B},\mathbf{B})=1, \\ g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{B})=g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{B})=0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1+N_2\mathbf{e}_2+N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N}=B_1\mathbf{e}_1+B_2\mathbf{e}_2+B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

**Theorem 3.1.** *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the parametric equations of $\gamma$ are*

$$\begin{aligned} x(s) &= \frac{\sin\mathfrak{P}e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1s+\mathfrak{C}_2]]+\mathfrak{C}_4, \\ y(s) &= \frac{\sin\mathfrak{P}e^{\cos\mathfrak{P}s+\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1s+\mathfrak{C}_2]]+\mathfrak{C}_5, \\ z(s) &= \cos\mathfrak{P}s+\mathfrak{C}_3, \end{aligned} \tag{3.4}$$

*where $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3,\mathfrak{C}_4,\mathfrak{C}_5$ are constants of integration.*

**Proof:** Assume that $\gamma$ a unit speed non-geodesic general helix. So, without loss of generality, we take the axis of $\gamma$ is parallel to the vector $\mathbf{e}_3$. Then,

$$g_{\mathfrak{sol}_3}(\mathbf{T},\mathbf{e}_3)=T_3=\cos\mathfrak{P},$$

where $\mathfrak{P}$ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T}=T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3. \tag{3.5}$$

On the other hand the tangent vector $\mathbf{T}$ is a unit vector, so the following condition is satisfied

$$T_1^2+T_2^2=1-\cos^2\mathfrak{P}.$$

Noting that $\cos^2\mathfrak{P}+\sin^2\mathfrak{P}=1$, we have

$$T_1^2+T_2^2=\sin^2\mathfrak{P}. \tag{3.6}$$

The general solution of (3.6) can be written in the following form

$$\begin{aligned} T_1 &= \sin\mathfrak{P}\cos\mathfrak{D}, \\ T_2 &= \sin\mathfrak{P}\sin\mathfrak{D}. \end{aligned}$$

So, substituting the components $T_1$, $T_2$ and $T_3$ in the equation (3.5), we have the following equation

$$\mathbf{T} = \sin\mathfrak{P}\cos\mathfrak{D}\mathbf{e}_1 + \sin\mathfrak{P}\sin\mathfrak{D}\mathbf{e}_2 + \cos\wp\mathbf{e}_3. \tag{3.7}$$

Also, without loss of generality, we take

$$\mathfrak{D} = \mathfrak{C}_1 s + \mathfrak{C}_2, \tag{3.8}$$

where $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathbb{R}$.

Thus (3.7) and (3.8), imply

$$\mathbf{T} = \sin\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]\mathbf{e}_1 + \sin\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]\mathbf{e}_2 + \cos\mathfrak{P}\mathbf{e}_3. \tag{3.9}$$

Using (2.4) in (3.9), we obtain

$$\mathbf{T} = (\sin\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]e^{-z}, \sin\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]e^{z}, \cos\mathfrak{P}).$$

Firstly, we have

$$\frac{dz}{ds} = \cos\mathfrak{P},$$

Integrating both sides, we have

$$z(s) = \cos\mathfrak{P}s + \mathfrak{C}_3,$$

where $\mathfrak{C}_3$ is constant of integration.

Secondly, we have

$$\frac{dx}{ds} = \sin\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]e^{-\cos\mathfrak{P}s - \mathfrak{C}_3}.$$

Also, integrating both sides, we have

$$x(s) = \frac{\sin\mathfrak{P}e^{-\cos\mathfrak{P}s - \mathfrak{C}_3}}{\mathfrak{C}_1^2 + \cos^2\mathfrak{P}}\left[-\cos\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] + \mathfrak{C}_1\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]\right] + \mathfrak{C}_4,$$

where $\mathfrak{C}_4$ is constant of integration.

Finallly, we obtain

$$\frac{dy}{ds} = \sin\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]e^{\cos\mathfrak{P}s + \mathfrak{C}_3}.$$

Since

$$y(s) = \frac{\sin\mathfrak{P}e^{\cos\mathfrak{P}s + \mathfrak{C}_3}}{\mathfrak{C}_1^2 + \cos^2\mathfrak{P}}\left[-\mathfrak{C}_1\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] + \cos\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]\right] + \mathfrak{C}_5,$$

where $\mathfrak{C}_5$ is constant of integration. This proves our assertion. Thus, the proof of theorem is completed. □

In terms of Eqs. (2.4) and (3.4), we may give:

**Theorem 3.2.** *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the equation of $\gamma$ is*

$$\begin{aligned} \gamma(s) \quad = \quad & [\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3}]\mathbf{e}_1 \\ & +[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}]\mathbf{e}_2 \\ & +[\cos\mathfrak{P}s+\mathfrak{C}_3]\mathbf{e}_3, \end{aligned} \tag{3.10}$$

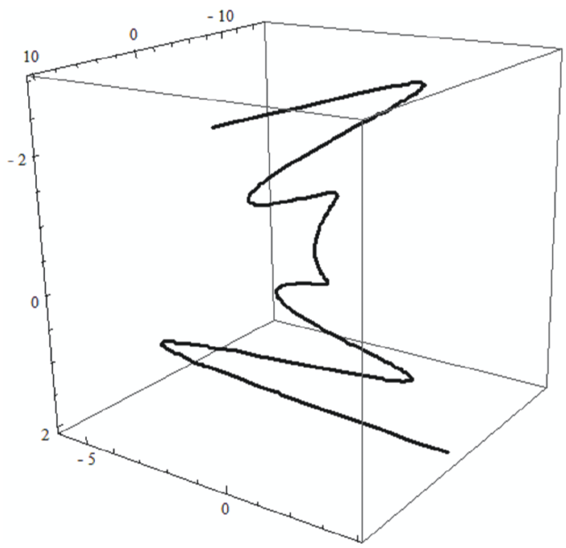
*where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.*

**Proof:** Suppose that $\gamma$ be a unit speed non-geodesic general helix. Using orthonormal basis of $\mathfrak{Sol}^3$, we have

$$\frac{\partial}{\partial x}=e^z\mathbf{e}_1, \quad \frac{\partial}{\partial y}=e^{-z}\mathbf{e}_2, \quad \frac{\partial}{\partial z}=\mathbf{e}_3. \tag{3.11}$$

Substituting (3.11) to (3.4), we have (3.10) as desired. This completes the proof. $\square$

We can use Mathematica in Theorem 3.1, yields

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