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Darboux Rotation Axis of Spacelike Biharmonic Helices with Timelike Normal in the Lorentzian $\mathbb{E}(1,1)$

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ABSTRACT: In this paper, we study Darboux rotation axis for spacelike biharmonic helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We obtain equation of Darboux vector of spacelike biharmonic helices in the Lorentzian $\mathbb{E}(1,1)$.

Key Words: Rigid motion, Darboux rotation, Bienergy.

Contents

1. Introduction

The object moves along the curve, let its intrinsic coordinate system keep itself aligned with the curve's Frenet frame. As it does so, the object's motion will be described by two vectors: a translation vector, and a rotation vector ω, which is an areal velocity vector: the Darboux vector.

Note that this rotation is kinematic, rather than physical, because usually when a rigid object moves freely in space its rotation is independent of its translation. The exception would be if the object's rotation is physically constrained to align itself with the object's translation, as is the case with the cart of a roller coaster.

In this paper, we study Darboux rotation axis for spacelike biharmonic helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We obtain equation of Darboux vector of spacelike biharmonic helices in the Lorentzian $\mathbb{E}(1,1)$.

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2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x,y,z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}(y+z),\ x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1},\ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1} \frac{\partial}{\partial x^2} + e^{-x^1} \frac{\partial}{\partial x^3} \right),\ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1} \frac{\partial}{\partial x^2} - e^{-x^1} \frac{\partial}{\partial x^3} \right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \tag{2.2}$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [10], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1} dx^2 + e^{x^1} dx^3\right)^2 + \left(e^{-x^1} dx^2 - e^{x^1} dx^3\right)^2, \tag{2.3}$$

where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1,\ g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\boldsymbol{\theta}^1 = dx^1,\ \boldsymbol{\theta}^2 = e^{-x^1} dx^2 + e^{x^1} dx^3,\ \boldsymbol{\theta}^3 = e^{-x^1} dx^2 - e^{x^1} dx^3.$$

Koszul's formula yields

$$\begin{aligned} \nabla_{\mathbf{X}_1} \mathbf{X}_1 &= 0, \quad \nabla_{\mathbf{X}_1} \mathbf{X}_2 = 0, \quad \nabla_{\mathbf{X}_1} \mathbf{X}_3 = 0, \\ \nabla_{\mathbf{X}_2} \mathbf{X}_1 &= -\mathbf{X}_3, \quad \nabla_{\mathbf{X}_2} \mathbf{X}_2 = 0, \quad \nabla_{\mathbf{X}_2} \mathbf{X}_3 = -\mathbf{X}_1, \\ \nabla_{\mathbf{X}_3} \mathbf{X}_1 &= -\mathbf{X}_2, \quad \nabla_{\mathbf{X}_3} \mathbf{X}_2 = -\mathbf{X}_1, \quad \nabla_{\mathbf{X}_3} \mathbf{X}_3 = 0. \end{aligned} \tag{2.5}$$

3. Spacelike Biharmonic Helices with Timelike Normal in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesic spacelike curve with timelike normal in the group of rigid motions $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields on the group of rigid motions $\mathbb{E}(1,1)$. along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T}+\tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ, τ is its torsion and

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N}) = -1,\ g(\mathbf{B},\mathbf{B}) = 1, \\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

Theorem 3.1. *(see [6]) $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic curve with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2+\tau^2 &= 1+2B_1^2, \\ \tau' &= -2N_1B_1. \end{aligned} \tag{3.3}$$

Theorem 3.2. *(see [6]) Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations of γ are*

$$x^1(s) = \sinh\wp\kappa s + a_1,$$

$$x^2(s) = \frac{\sqrt{1+\sinh^2\wp}\,e^{\sinh\wp\kappa s+a_1}}{2(\Lambda^2+\sinh^2\wp)}(\sinh\wp+\Lambda)\left[\cos\left[\Lambda\kappa s+\Upsilon\right]-\sin\left[\Lambda\kappa s+\Upsilon\right]\right]+a_2,$$

$$x^3(s) = \frac{\sqrt{1+\sinh^2\wp}\,e^{-\sinh\wp\kappa s-a_1}}{2(\Lambda^2+\sinh^2\wp)}(\sinh\wp-\Lambda)\left[\cos\left[\Lambda\kappa s+\Upsilon\right]-\sin\left[\Lambda\kappa s+\Upsilon\right]\right]+a_3,$$

where Λ, Υ, a_1, a_2, a_3 are constants of integration.

4. Darboux Rotation Axis of Spacelike Biharmonic Helices with Timelike Normal in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Using Frenet equations form a rotation motion with Darboux vector,

$$\mathbf{D} = -\tau \mathbf{T} + \kappa \mathbf{B}. \tag{4.1}$$

From above equation, momentum rotation vector is expressed as follows:

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= \mathbf{D} \times \mathbf{T}, \\ \nabla_{\mathbf{T}} \mathbf{N} &= \mathbf{D} \times \mathbf{N}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= \mathbf{D} \times \mathbf{B}. \end{aligned}$$

Darboux rotation of Frenet frame can be separated into two rotation motions : $\mathbf{T}$ tangent vector rotates with a κ angular speed round $\mathbf{B}$ binormal vector, that is

$$\nabla_{\mathbf{T}} \mathbf{T} = (\kappa \mathbf{B}) \times \mathbf{T}$$

and $\mathbf{B}$ binormal vector rotates with a τ angular speed round $\mathbf{T}$ tangent vector, that is

$$\nabla_{\mathbf{T}} \mathbf{B} = (-\tau \mathbf{T}) \times \mathbf{B}.$$

Lemma 4.1. $\mathbf{D}$ *vector rotates with zero angular speed round* $\mathbf{N}$ *principal normal for spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$.

Proof: We assume that $\mathbf{D}$ vector rotates round $\mathbf{N}$ principal normal of γ.

So, by differentiating of the formula (4.1), we get

$$\nabla_{\mathbf{T}} \mathbf{D} = -\tau' \mathbf{T} + \kappa' \mathbf{B}.$$

Since Theorem 3.2, we immediately arrive at

$$\nabla_{\mathbf{T}} \mathbf{D} = 0.$$

In terms of above equation, we may give:

$$g\left(\nabla_{\mathbf{T}} \mathbf{D}, \nabla_{\mathbf{T}} \mathbf{D}\right) = 0.$$

This concludes the proof of Lemma.

□

Hence, we have the following theorem.

Theorem 4.2. *Let* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is a non geodesic spacelike biharmonic helix with timelike normal in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$. *Then, Darboux vector of* γ *is constant vector.*

Proof: Using Lemma 4.1, we immediately arrive at $\mathbf{D}$ is constant vector. □

Theorem 4.3. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$is a non geodesic spacelike biharmonic helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the equation of Darboux vector of γ is*

$$\begin{aligned} \mathbf{D} = & [-\tau \sinh\wp + \cosh^2\wp\cos^2[\Lambda\kappa s + \Upsilon](-\sinh\wp + \frac{1}{\Lambda\kappa}) \\ & + \cosh^2\wp\sin^2[\Lambda\kappa s + \Upsilon](\sinh\wp + \frac{1}{\Lambda\kappa})]\mathbf{X}_1 \qquad (4.2) \\ & +[-\tau\cosh\wp\cos[\Lambda\kappa s + \Upsilon] + \cosh\wp\sinh\wp\cos[\Lambda\kappa s + \Upsilon](-\sinh\wp + \frac{1}{\Lambda\kappa}) \\ & +2\left(\cosh^2\wp\right)^{\frac{3}{2}}\sin^2[\Lambda\kappa s + \Upsilon]\cos[\Lambda\kappa s + \Upsilon]]\mathbf{X}_2 \\ & +[-\tau\cosh\wp\sin[\Lambda\kappa s + \Upsilon] - \cosh\wp\sinh\wp\sin[\Lambda\kappa s + \Upsilon](\sinh\wp + \frac{1}{\Lambda\kappa}) \\ & +2\left(\cosh^2\wp\right)^{\frac{3}{2}}\cos^2[\Lambda\kappa s + \Upsilon]\sin[\Lambda\kappa s + \Upsilon]]\mathbf{X}_3. \end{aligned}$$

Proof: Using Theorem 3.1, the tangent vector of γ can be written in the following form:

$$\mathbf{T} = \sinh\wp\mathbf{X}_1 + \sqrt{1+\sinh^2\wp}\cos[\Lambda\kappa s + \Upsilon]\mathbf{X}_2 + \sqrt{1+\sinh^2\wp}\sin[\Lambda\kappa s + \Upsilon]\mathbf{X}_3. \qquad (4.3)$$

By a straightforward calculation, we get

$$\begin{aligned} \mathbf{B} = & \mathbf{T}\times\mathbf{N} = [\frac{\left(1+\sinh^2\wp\right)}{\kappa}\cos^2[\Lambda\kappa s + \Upsilon](-\sinh\wp + \frac{1}{\Lambda\kappa}) \\ & +\frac{\left(1+\sinh^2\wp\right)}{\kappa}\sin^2[\Lambda\kappa s + \Upsilon](\sinh\wp + \frac{1}{\Lambda\kappa})]\mathbf{X}_1 \\ & +[\frac{\sqrt{1+\sinh^2\wp}}{\kappa}\sinh\wp\cos[\Lambda\kappa s + \Upsilon](-\sinh\wp + \frac{1}{\Lambda\kappa}) \qquad (4.4) \\ & +2\frac{\left(1+\sinh^2\wp\right)^{\frac{3}{2}}}{\kappa}\sin^2[\Lambda\kappa s + \Upsilon]\cos[\Lambda\kappa s + \Upsilon]]\mathbf{X}_2 \\ & [-\frac{\sqrt{1+\sinh^2\wp}}{\kappa}\sinh\wp\sin[\Lambda\kappa s + \Upsilon](\sinh\wp + \frac{1}{\Lambda\kappa}) \\ & +2\frac{\left(1+\sinh^2\wp\right)^{\frac{3}{2}}}{\kappa}\cos^2[\Lambda\kappa s + \Upsilon]\sin[\Lambda\kappa s + \Upsilon]]\mathbf{X}_3. \end{aligned}$$

If we substitute the equations (4.3) and (4.4) in the equation (4.1), we have (4.2), which it completes the proof. □

We put

$$\mathbf{E} = \frac{\mathbf{D}}{\|\mathbf{D}\|}.$$

Corollary 4.4. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.Then,* **E** *is constant vector.*

According to the second Frenet formula, we have

$$\nabla_{\mathbf{T}}\mathbf{N} = -\left|g\left(\mathbf{D},\mathbf{D}\right)\right|^{\frac{1}{2}}\left(\mathbf{E}\times\mathbf{N}\right)$$

or

$$\nabla_{\mathbf{T}}\mathbf{N} = -\sqrt{1+2B_1^2}\left(\mathbf{E}\times\mathbf{N}\right).$$

Furthermore, we put

$$\mathbf{U} = \mathbf{E}\times\mathbf{N}.$$

On the other hand, from above corollary we have

$$\nabla_{\mathbf{T}}\mathbf{U} = -\sqrt{1+2B_1^2}\mathbf{N}$$

Since,

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{E} &= 0,\\
\nabla_{\mathbf{T}}\mathbf{N} &= -\sqrt{1+2B_1^2}\left(\mathbf{E}\times\mathbf{N}\right),\\
\nabla_{\mathbf{T}}\mathbf{U} &= -\sqrt{1+2B_1^2}\mathbf{N}.
\end{aligned}$$

Thus, the vectors $\mathbf{N}$, $\mathbf{E}\times\mathbf{N}$, $\mathbf{E}$ define a rotation motion together the rotation vector,

$$\mathbf{D}_1 = \sqrt{1+2B_1^2}\mathbf{E}.$$

Corollary 4.5. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, momentum rotation vector is expressed as follows:*

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{N} &= \mathbf{D}_1\times\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{U} &= \mathbf{D}_1\times\mathbf{U}.
\end{aligned}$$

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