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The Rate of Sectional entire sequence spaces

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ABSTRACT: The space $(\Gamma_s)_\pi$ is introduced. This paper deveoted a study of the general properties of sectional rate space $(\Gamma_s)_\pi$ of Γ.

Key Words: Rate space, entire sequences, analytic sequences, β dual..

Contents

1. Introduction

A complex sequence, whose k^{th} terms x_k is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequence. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k |x_k|^{1/k} < \infty$. A vector space of all analytic sequence will be denoted by Λ. A sequence $x = \{x_k\}$ is called entire if $lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of entire sequence denoted by Γ. Kizmaz [22] defined the following difference squence spaces

$$Z(\Delta) = \{x = (x_k) : \Delta x \in Z\}$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x = (\Delta x)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$ and showed that these are Banach space with norm $\|x\| = |x_1| + \|\Delta x\|_\infty$. Later on Et and Colak [23] generalized the notion as follows :
Let $m \in \mathbb{N}$, $Z(\Delta^m) = \{x = (x_k) : \Delta^m x \in Z\}$ for $Z = \ell_\infty, c, c_0$ where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k)_{k=1}^{\infty} = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)_{k=1}^{\infty}$.
The generalized differences has the following binomial representation:

$$\Delta^m x_k = \sum_{\gamma=0}^{m} (-1)^\gamma \binom{m}{\gamma} x_{k+\gamma},$$

They proved that these are Banach spaces with the norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$$

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Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$ $\delta^{(n)} = (0, 0, ..., 1, 0, 0, ...)$, 1 in the n^{th} place and zero's else where and $s^k = \{0, 0, 0, \cdots, 1, -1, 0, 0, \cdots\}$, 1 in the n^{th} place and -1 in the $(n+1)^{th}$ place and zero's else where.

If X is a sequence space, we define

(i)$X^{'}$ = the continuous dual of X.

(ii)$X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \, for\, each\, x \in X\}$;

(iii)$X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \, is\, convergent,\, for\, each\, x \in X\}$;

(iv)$X^\gamma = \left\{a = (a_k) : \overset{sup}{n} \left|\sum_{k=1}^{n} a_k x_k\right| < \infty, \, for\, each\, x \in X\right\}$;

(v)Let X be an FK-space$\supset \phi$. Then $X^f = \left\{f(\delta^{(n)}) : f \in X^{'}\right\}$.

$X^\alpha, X^\beta, X^\gamma$ are called the $\alpha-$(or Kö the-T öeplitz)dual of X, $\beta-$ (or generalized Kö the-T öeplitz)dual of X, $\gamma-$dual of X. Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$ then $Y^\mu \subset X^\mu$, for $\mu = \alpha, \beta, or\, \gamma$.

An FK-space(Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k\, (k = 1, 2, ...)$ are continuous. We recall the following definitions[see [14]]. An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coodinate projections are continuous. An metric space (X, d) is said to have AK (or sectional convergence) if and only if $d\left(x^{(n)}, x\right) \to 0$ as $n \to \infty$[see [17]]. The space is said to have AD or be an AD space if ϕ is dense in X. We note that AK implies AD by [14].

2. Definitions and Preliminaries

Throughout the paper w, Γ and Λ denote the spaces of all entire and bounded sequence respectively. Let t denote the sequence with $t = |x_k|^{1/k}$ for all $k \in \mathbb{N}$. Define the sets: $\Gamma = \{x \in w : t_k \to 0\, as\, k \to \infty\}$ and $\Lambda = \{x \in w : sup_k t_k < \infty\}$. The spaces Γ and Λ are metric space with the metric

$$d\,(x, y) = sup_k \left\{|x_k - y_k|^{1/k} : k = 1, 2, 3, \cdots\right\}$$

Let $(\Gamma_s)_\pi = \left\{x = (x_k) \in w : \xi = \left(\frac{\xi_k}{\pi_k}\right) \in \Gamma\right\}$; where $\xi_k = x_1 + x_2 + \cdots + x_k$. Let $(\Lambda_s)_\pi = \left\{x = (x_k) \in w : \eta = \left(\frac{\eta_k}{\pi_k}\right) \in \Lambda\right\}$; where $\eta_k = y_1 + y_2 + \cdots + y_k$. The spaces $(\Gamma_s)_\pi$ and $(\Lambda_s)_\pi$ are metric space with the metric

$$d\,(x, y) = sup_k \left\{\left|\frac{\xi_k - \eta_k}{\pi_k}\right|^{1/k} : k = 1, 2, 3, \cdots\right\}$$

Let $\sigma\,(\Gamma)$ denote the vector space of all sequences $x = (x_k)$ such that $\left(\frac{\xi_k}{k}\right)$ is entire sequence. We recall that cs_0 denotes the vector space of all sequences $x = (x_k)$ such that (ξ_k) is a null sequence.

Lemma 2.1 *(see ([17], Theorem 7.2.7)) Let X be an FK-space$\supset \phi$. Then (i) $X^\gamma \subset X^f$. (ii) If X has AK, $X^\beta = X^f$. (iii) If X has AD, $X^\beta = X^\gamma$.*

Remark 2.2 $x = (x_k) \in \sigma(\Gamma_s) \Leftrightarrow \left(\frac{\xi_k}{\pi_k k}\right) \in \Gamma \Leftrightarrow \left|\frac{\xi_k}{\pi_k k}\right|^{1/k} \to 0\, as\, k \to \infty \Leftrightarrow \left|\frac{\xi_k}{\pi_k}\right|^{1/k} \to 0\, as\, k \to \infty,$ *because* $k^{1/k} \to 1\, as\, k \to \infty \Leftrightarrow x = (x_k) \in (\Gamma_s)_\pi$. *Hence* $(\Gamma_s)_\pi = \sigma(\Gamma_s)$, *the cesàro space of order 1.*

3. Main Results

Proposition 3.1 $(\Gamma_s)_\pi \subset \Gamma_\pi$

Proof: Let $x \in (\Gamma_s)_\pi$
$\Rightarrow \xi \in \Gamma_\pi$

$$\left|\frac{\xi_k}{\pi_k}\right|^{1/k} \to 0\, as\, k \to \infty \tag{3.1}$$

But $\frac{x_k}{\pi_k} = \frac{\xi_k}{\pi_k} - \frac{\xi_{k-1}}{\pi_{k-1}}$.
$\left|\frac{x_k}{\pi_k}\right|^{1/k} \le \left|\frac{\xi_k}{\pi_k} - \frac{\xi_{k-1}}{\pi_{k-1}}\right|^{1/k} \le \left|\frac{\xi_k}{\pi_k}\right|^{1/k} + \left|\frac{\xi_{k-1}}{\pi_{k-1}}\right|^{1/k} \to 0\, as\, k \to \infty$ by using (3.1)
Therefore $\left|\frac{x_k}{\pi_k}\right|^{1/k} \to 0\, as\, k \to \infty$
$\Rightarrow x \in \Gamma_\pi$. Hence $(\Gamma_s)_\pi \subset \Gamma_\pi$.
Note: The above inclusions is strict. Take the sequence $\delta^{(1)} \in \Gamma_\pi$. We have

$\left(\frac{\xi_1}{\pi_1}\right) = 1$

$\left(\frac{\xi_2}{\pi_2}\right) = 1 + 0 = 1$

$\left(\frac{\xi_3}{\pi_3}\right) = 1 + 0 + 0 = 1$

$\vdots$

$\left(\frac{\xi_k}{\pi_k}\right) = 1 + 0 + 0 + \cdots + = 1$

$\to k - terms \to$

and so on. Now $\left(\left|\frac{\xi_k}{\pi_k}\right|^{1/k}\right) = 1$ for all k. Hence $\left\{\left|\frac{\xi_k}{\pi_k}\right|^{1/k}\right\}$ does not tend to zero as $k \to \infty$. So $\delta^{(1)} \notin (\Gamma_s)_\pi$. Thus the inclusion $(\Gamma_s)_\pi \subset \Gamma_\pi$ is strict. This completes the proof. □

Proposition 3.2 $(\Gamma_s)_\pi$ *is a linear space over the field* $\mathbb{C}$ *of complex numbers.*

Proof: It is easy. Therefore omit the proof. □

Proposition 3.3 $\Lambda_\pi \subset (\Gamma_s)_\pi \subset \Lambda_\pi(\Delta)$

Proof: Step 1. By Proposition 3.1, we have $(\Gamma_s)_\pi \subset \Gamma_\pi$. Hence $(\Gamma_\pi)^\beta \subset [(\Gamma_s)_\pi]^\beta$. But $(\Gamma_\pi)^\beta = \Lambda_\pi$. Therefore

$$\Lambda_\pi \subset [(\Gamma_s)_\pi]^\beta \tag{3.2}$$

Step2: Let $y = (y_k) \in [(\Gamma_s)_\pi]^\beta$. Consider $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in (\Gamma_s)_\pi$. Take $x = \delta^n - \delta^{n+1} = (0, 0, 0, \cdots, \pi, -\pi, 0, 0, \cdots)$
$n^{th}(n+1)^{th} place$
where, for each fixed $n = 1, 2, 3, \cdots ; \delta^{(n)} = (0, 0, \cdots \pi, 0, \cdots), \pi$ in the n^{th} place and zero's elsewhere. Then $f\left(\delta^n - \delta^{n+1}\right) = y_n - y_{n+1}$. Hence
$\left|\frac{y_n}{\pi_n} - \frac{y_{n+1}}{\pi_{n+1}}\right| = \left|f\left(\delta^n - \delta^{n+1}\right)\right| \le \|f\| \, d\left(\left(\delta^n - \delta^{n+1}\right), 0\right) \le \|f\| \cdot 1.$
So, $\left|\frac{y_n}{\pi_n} - \frac{y_{n+1}}{\pi_{n+1}}\right|$ is bounded. Consequently $\left|\frac{y_n}{\pi_n} - \frac{y_{n+1}}{\pi_{n+1}}\right| \in \Lambda_\pi$ That is $\left(\frac{y_n}{\pi_n}\right) \in \Lambda_\pi(\Delta)$. But $y = (y_n)$ is originally in $[(\Gamma_s)_\pi]^\beta$. Therefore

$$[(\Gamma_s)_\pi]^\beta \subset \Lambda_\pi(\Delta) \tag{3.3}$$

From (3.2) and (3.3) we conclude that $\Lambda_\pi \subset [(\Gamma_s)_\pi]^\beta \subset \Lambda_\pi(\Delta)$. This completes the proof. □

Proposition 3.4 *The $\beta-$ dual space of $(\Gamma_s)_\pi$ is Λ_π*

Proof: Step1. Let $y = (y_k)$ be an arbitray point in $[(\Gamma_s)_\pi]^\beta$. If y is not in Λ_π, then for each natural number n, we can find an index $k(n)$ such that

$$\left(\left|\frac{y_{k(n)}}{\pi_{k(n)}}\right|^{1/k(n)}\right) > \frac{1}{n}, \, (n = 1, 2, 3, \cdots)$$

Define $x = (x_k)$ by

$$\left(\frac{x_k}{\pi_k}\right) = 1/n^k, \, for\, k = k(n); \text{and } \left(\frac{x_k}{\pi_k}\right) = 0 \text{ otherwise}$$

Then x is in Γ_π, but for infinitely many k,

$$\left(\frac{x_k y_k}{\pi_k}\right) > 1 \tag{3.4}$$

Consider the sequence $z = \{z_k\}$, where $z_1 = \frac{x_1}{\pi_1} - s$ with $s = \sum \frac{x_k}{\pi_k}$; and $z_k = \frac{x_k}{\pi_k} \, (k = 2, 3, \cdots)$ Then z is a point of Γ_π. Also $\left(\frac{x_k}{\pi_k}\right) = 0$ Hence z is in $(\Gamma_s)_\pi$. But by the equation(3.4), $\sum\left(\frac{z_k x_k}{\pi_k}\right)$ does not converge. Thus the sequence y would not to be in $[(\Gamma_s)_\pi]^\beta$. This contradiction proves that

$$[(\Gamma_s)_\pi]^\beta \subset \Lambda_\pi \tag{3.5}$$

Step2. By (3.2) of Proposition 3.3, we have

$$\Lambda_\pi \subset [(\Gamma_s)_\pi]^\beta \tag{3.6}$$

From (3.5) and (3.6) it follows that the $\beta-$ dual space of $[(\Gamma_s)_\pi]^\beta$ is Λ_π This completes the proof. □

Proposition 3.5 *Suppose that* $x \in \Gamma_\pi$, *then* $x \in [(\Gamma_s)_\pi] \Leftrightarrow \sum_{k=1}^\infty \frac{x_k}{\pi_k} = 0$

Proof: Let

$$\sum_{k=1}^{\infty} \frac{x_k}{\pi_k} = 0 \tag{3.7}$$

Note that $x \in \Gamma_\pi$ implies that the sum on the left hand side of (3.7) must exists. Denote the sum by s. Clearly $\left(\frac{\epsilon_k}{\pi_k}\right) \to s\, as\, k \to \infty$. Hence it follows that if $s \neq 0$ then $x \notin \Gamma_\pi$. Assume that $s = 0$ then

$$\left(\frac{\epsilon_k}{\pi_k}\right) = -\sum_{\gamma=k+1}^{\infty} \left(\frac{x_\nu}{\pi_\nu}\right) \tag{3.8}$$

Let $\epsilon > 0$ be given with $\epsilon < 1/2$. There is a k_0 such that $\left(\frac{x_\nu}{\pi_\nu}\right)^{1/\nu} < \epsilon$ for all $\nu \geq k_0$ equation (3.8) gives us the results for $k \geq k_0$. Now $\left|\frac{\epsilon_k}{\pi_k}\right| < \sum_{\nu=k+1}^\infty \epsilon^\nu = \frac{\epsilon^{k+1}}{1-\epsilon} < \epsilon_k$, since $\epsilon < 1/2$. Thus $\left|\frac{\epsilon_k}{\pi_k}\right|^{1/k} < \epsilon$. Consequently $x \in [(\Gamma_s)_\pi]$. This completes the proof. □

Proposition 3.6 $[(\Gamma_s)_\pi]$ *is closed in* Γ_π *in the metric toplogy of* Γ_π

Proof: If $x \in \Gamma_\pi - [(\Gamma_s)_\pi]$ then $s = 0$. Let $\epsilon > 0$ be given. If $d(x, y) < \epsilon$, then $\left|\sum_{k=1}^\infty \frac{x_k}{\pi_k} - \sum_{k=1}^\infty \frac{y_k}{\pi_k}\right| \leq \sum_{k=1}^\infty \left|\frac{x_k}{\pi_k} - \frac{y_k}{\pi_k}\right| \leq \epsilon^k = \frac{\epsilon}{1-\epsilon}$.

This can be made arbitrary small by the choice of ϵ. Thus however $d(x, y)$ is sufficiently small, we have

$$\textstyle\sum_{k=1}^\infty \frac{y_k}{\pi_k} \neq 0.$$

So that by Theorem 3.5 $y \notin [(\Gamma_s)_\pi]$. Accordingly $y \in \Gamma_\pi - [(\Gamma_s)_\pi]$ for each y in the open ball $B(x, \epsilon)$ with centre x and radius ϵ. Hence $x \in B(x, \epsilon) \subset \Gamma_\pi - [(\Gamma_s)_\pi]$. Therefore each point x of $\Gamma_\pi - [(\Gamma_s)_\pi]$ is an interior point $\Gamma_\pi - [(\Gamma_s)_\pi]$. In other words $\Gamma_\pi - [(\Gamma_s)_\pi]$ is open in Γ. Consequently $[(\Gamma_s)_\pi]$ is closed in Γ_π. This completes the proof. □

Proposition 3.7 $[(\Gamma_s)_\pi]$ *is a complete metric space*

Proof: The metric d for Γ_π is given by $d(x, y)$. The metric d for Γ_π may be restricted to $[(\Gamma_s)_\pi]$ then $[(\Gamma_s)_\pi, d]$ is a metric space. Where

$$d(x,y) = sup_k \left\{ \left| \frac{x_k - y_k}{\pi_k} \right|^{1/k} : k = 1,2,3,\cdots \right\}$$

But by Theorem (3.5), $[(\Gamma_s)_\pi, d]$ is a closed subspace of Γ_π. Also Γ_π is a complete metric space [see [1]]. Hence $(\Gamma_s)_\pi$ is a complete metric space. □

Proposition 3.8 $[(\Gamma_s)_\pi, d]$ *is separable.*

Proof: Write
$\delta^1 = \{\pi, 0, 0\cdots,\}$;
$\delta^2 = \{0, \pi, 0, \cdots,\}$;

$\vdots$

$\delta^k = \{0, 0, 0\cdots, \pi\}$; π in the k^{th} place and 0 elsewhere. Where π is the constant sequence $\{\pi, \pi, \cdots,\}$ with $\pi > 0$.

Then $A = \{\sigma^1, \sigma^2, \cdots\}$ is a Schauder basis for Γ_π correspondingly we have,
$\rho^1 = \{\pi, -\pi, 0, \cdots,\}$;
$\rho^2 = \{0, \pi, -\pi, 0, \cdots,\}$;

$\vdots$

$\rho^k = \{0, 0, 0\cdots, \pi, -\pi\}$; π in the k^{th} place and $-\pi$ in the $(k+1)^{th}$ place and zero elsewhere. Then $B = \{\rho^1, \rho^2, \rho^3, \cdots,\}$ is Schauder basis in $[(\Gamma_s)_\pi, d]$. Hence $[(\Gamma_s)_\pi, d]$ is separable. This completes the proof. □

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