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Application of Chybeshev Polynomials in Factorizations of Balancing and Lucas-Balancing Numbers

Prasanta Kumar Ray

ABSTRACT: In this paper, with the help of orthogonal polynomials especially Chybeshev polynomials of first and second kind, number theory and linear algebra intertwined to yield factorization of balancing and Lucas-balancing numbers.

Key Words: Balancing numbers, balancers, Lucas-balancing numbers, triangular numbers.

Contents

1. Introduction

As usual, see [1], the balancing number n is defined by the solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

where r is the balancer corresponding to the balancing number n. The first few balancing numbers are 1, 6, 35 with corresponding balancers 0, 2, 14. If B_n is the n^{th} balancing number, the recurrence relation for balancing numbers is given by

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 2, \tag{1.1}$$

with $B_1 = 1, B_2 = 6$.
In [1] it is shown that, if n is a balancing number, n^2 is a triangular number, that is, $8n^2 + 1$ is a perfect square and for all n, $\sqrt{8n^2+1}$ generates a sequence called as the sequence of Lucas-balancing numbers [5], whose first few terms are given by 1, 3 and 17 and if C_n is the n^{th} Lucas-balancing number, its recurrence relation is given by

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2, \tag{1.2}$$

with $C_1 = 3, C_2 = 17$.
In the recent years many number theorists from all over the world are taking interest in this beautiful number system. Liptai [2] proved that the only Fibonacci number

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in the sequence of balancing numbers is 1. In [3], he also proved that there is no Lucas number in the sequence of balancing numbers. Balancing numbers and its related sequences are available in the literature. Interested reader may follow [4], [6], [7].
In this paper, we observe that, with the help of orthogonal polynomials, number theory and linear algebra intertwined to yield factorization of balancing and Lucas-balancing numbers. In section 2 and 3 we derive the following factorization of these numbers:

$$B_n = \prod_{1\leq k\leq n-1} (6 - 2\cos\frac{k\pi}{n}) \tag{1.3}$$

$$C_n = \frac{1}{2}[\prod_{1\leq k\leq n} (6 - 2\cos\frac{(2k-1)\pi}{2n})] \tag{1.4}$$

In order to derive (1.3) and (1.4) we present the following theorem whose proof is included for completeness.

Theorem 1.1 *If the sequence of tridiagonal matrices $\{A_n, n = 1, 2, \cdots\}$ is of the form*

$$A_n = \begin{pmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & A_{23} & & \\ & A_{32} & A_{33} & \ddots & \\ & & \ddots & \ddots & A_{(n-1)n} \\ & & & A_{n(n-1)} & A_{nn} \end{pmatrix},$$

then the successive determinant of A_n are given by the recursive formulas:

$$\begin{aligned} det(A_1) &= A_{11} \\ det(A_2) &= A_{11}A_{22} - A_{12}A_{21} \\ det(A_n) &= A_{nn}\ det(A_{n-1}) - A_{(n-1)n}A_{n(n-1)}\ det(A_{n-2}). \end{aligned}$$

Proof. Using Induction one can easily check that the theorem is true for $n = 1, 2$ and 3 and assume that it is true for all k, $3 \leq k \leq n$, that is

$$det(A_k) = A_{kk}\ det(A_{k-1}) - A_{(k-1)k}A_{k(k-1)}\ det(A_{k-2}).$$

Now,

$$det(A_{k+1}) = det\begin{pmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & A_{23} & & \\ & A_{32} & A_{33} & \ddots & \\ & & \ddots & \ddots & A_{k(k+1)} \\ & & & A_{(k+1)k} & A_{(k+1)(k+1)} \end{pmatrix}$$

$$= A_{(k+1)(k+1)}\ det(A_k) - A_{k(k+1)}\ det\begin{pmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & A_{23} & & \\ & A_{32} & A_{33} & \ddots & \\ & & \ddots & \ddots & A_{(k-1)k} \\ & & & A_{k(k-1)} & A_{(k+1)k} \end{pmatrix}$$

$$= A_{(k+1)(k+1)}\ det(A_k) - A_{k(k+1)}A_{(k+1)k}\ det(A_{k-1}).$$

Thus the theorem is true for all natural number n.

2. Factorization of Balancing Numbers

In order to derive the factorization of balancing numbers (1.3), let us introduce the sequence of matrices $\{D_n, n = 1, 2, \cdots\}$ where D_n is am $n \times n$ tridiagonal matrix with entries $d_{kk} = 6,\ \ 1 \le k \le n$ and $d_{(k-1)k} = -i, d_{k(k-1)} = i,\ \ 2 \le k \le n$, where $i = \sqrt{-1}$. That is

$$D_n = \begin{pmatrix} 6 & -i & & & \\ i & 6 & -i & & \\ & i & 6 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 6 \end{pmatrix},$$

By virtue of Theorem 1.1, we find

$$det(D_1) = 6$$
$$det(D_2) = 36 + i^2 = 35$$
$$det(D_n) = 6\ det(D_{n-1}) - det(D_{n-2}),$$

which is nothing but the sequence of balancing numbers starting with B_2. Thus,

$$B_n = det(D_{n-1}), \quad n \ge 2. \tag{2.1}$$

Since the determinant of a matrix can be found by taking the product of its eigenvalues, we will now find the spectrum of D_n in order to find an alternate formulation for $det(D_n)$.

Let us introduce another sequence of matrices $\{S_n, n = 1, 2, \cdots\}$ where S_n is an $n \times n$ tridiagonal matrix with entries $s_{kk} = 0, \quad 1 \leq k \leq n$ and $s_{(k-1)k} = -i, s_{k(k-1)} = i, \quad 2 \leq k \leq n$. That is,

$$S_n = \begin{pmatrix} 0 & -i & & & \\ i & 0 & -i & & \\ & i & 0 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 0 \end{pmatrix}.$$

Clearly $D_n = 6I + S_n$, where I be the identity matrix same order as S_n. Let $\lambda_k, k = 1, 2, 3 \cdots, n$, be the eigenvalues of S_n with corresponding eigenvectors X_k. Then for each j,

$$\begin{aligned} D_n X_j &= [6I + S_n]X_j \\ &= 6IX_j + S_n X_j \\ &= 6X_j + \lambda_j X_j \\ &= (6 + \lambda_j)X_j. \end{aligned}$$

Thus $\delta_k = 6 + \lambda_k, \quad k = 1, 2, \cdots, n$, be the eigenvalues of D_n. Therefore,

$$det(D_n) = \prod_{1 \leq k \leq n} (6 + \lambda_k), \quad n \geq 1. \tag{2.2}$$

In order to find $\lambda_k, k = 1, 2 \cdots, n$, we recall that each λ_k is zero of the characteristic polynomial $p_n(\lambda) = det(S_n - \lambda I)$.
Since $S_n - \lambda I$ is a tridiagonal matrix we have,

$$S_n - \lambda I = \begin{pmatrix} -\lambda & -i & & & \\ i & -\lambda & -i & & \\ & i & -\lambda & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & -\lambda \end{pmatrix}.$$

Using Theorem 1.1, we get the following recursive formula for the characteristic polynomials:

$$\begin{aligned} p_1(\lambda) &= -\lambda \\ p_2(\lambda) &= \lambda^2 - 1 \\ p_n(\lambda) &= -\lambda p_{n-1}(\lambda) - p_{n-2)}(\lambda). \end{aligned}$$

This family of polynomials can be transformed into another family $\{M_n, n \geq 1\}$ by the transformation $\lambda = -2x$ to get,

$$\begin{aligned} M_1(x) &= 2x \\ M_2(x) &= 4x^2 - 1 \\ M_n(x) &= 2xM_{n-1}(x) - M_{n-2}(x). \end{aligned}$$

We observe that the family $\{M_n, n \geq 1\}$ is the set of Chebyshev polynomials of second kind. It is well known that for $x = \cos\theta$, the Chebyshev polynomials of the second kind can be written as

$$M_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}$$

which when equal to zero gives

$$\theta_k = \frac{\pi k}{n+1}, \quad k = 1, 2, \cdots n.$$

Thus,

$$\begin{aligned} x_k &= \cos\theta_k \\ &= \cos\frac{\pi k}{n+1}, \quad k = 1, 2, \cdots n. \end{aligned}$$

Now applying the transformation $\lambda = -2x$, the eigenvalues of S_n are given by

$$\lambda_k = -2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \cdots n. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we get

$$B_{n+1} = det(D_n) = \prod_{1\leq k\leq n} (6 - 2\cos\frac{k\pi}{n}), \quad n \geq 1,$$

which is identical to the factorization (1.3).

3. Factorization of Lucas-Balancing Numbers

In a similar manner we can derive (1.4) by considering the sequence of matrices $\{E_n, n = 1, 2, \cdots\}$ where E_n is an $n \times n$ tridiagonal matrix with entries $e_{11} = 3, e_{kk} = 6,\ 2 \leq k \leq n$ and $e_{(k-1)k} = -i, s_{k(k-1)} = i, \quad 2 \leq k \leq n$. That is,

$$E_n = \begin{pmatrix} 3 & -i & & & \\ i & 6 & -i & & \\ & i & 6 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 6 \end{pmatrix}.$$

Again using Theorem 1.1, we obtain

$$\begin{aligned} det(E_1) &= 3 \\ det(E_2) &= 18 + i^2 = 17 \\ det(E_n) &= 6\ det(E_{n-1}) - det(E_{n-2}). \end{aligned}$$

We observe that each member in this sequence is a Lucas-balancing number. Thus, we get

$$C_n = det(E_n), \quad n \geq 1. \tag{3.1}$$

If e_j is the j^{th} column of the identity matrix I, we see that $det(I + e_1e_1^T) = 2$. Therefore,we may write

$$det(E_n) = \frac{1}{2}det[(I + e_1e_1^T)E_n]. \tag{3.2}$$

Also we observe that the right hand side of (3.2) can be expressed as

$$\frac{1}{2}det[(I + e_1e_1^T)E_n] = \frac{1}{2}det[6I + S_n - ie_1e_2^T]$$

where S_n is the matrix defined earlier.
If $\alpha_k, \quad k = 1, 2, 3 \cdots, n$, be the eigenvalues of $S_n - ie_1e_2^T$ with corresponding eigenvectors Y_k, then for each j,

$$\begin{aligned} [6I + S_n - ie_1e_2^T]Y_j &= 6IY_j + (S_n - ie_1e_2^T)Y_j \\ &= 6Y_j + \alpha_j Y_j \\ &= (6 + \alpha_j)Y_j. \end{aligned}$$

Therefore,

$$\frac{1}{2}det[6I + S_n - ie_1e_2^T] = \frac{1}{2}\prod_{1\leq k\leq n}(6 + \alpha_k), \quad n \geq 1. \tag{3.3}$$

In order to find $\alpha_k^{'}s$, we recall that each α_k is a zero of the characteristic polynomial $q_n(\alpha) = det(S_n - ie_1e_2^T - \alpha I)$. Since $det(I - \frac{1}{2}e_1e_1^T) = \frac{1}{2}$, we can express the characteristic polynomial as

$$\begin{aligned} q_n(\alpha) &= 2det[(I - \frac{1}{2}e_1e_1^T)(S_n - ie_1e_2^T - \alpha I)] \\ &= 2det\begin{pmatrix} \frac{-\alpha}{2} & -i & & & \\ i & -\alpha & -i & & \\ & i & -\alpha & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & -\alpha \end{pmatrix}. \end{aligned}$$

Since $q_n(\alpha)$ is the twice of a tridiagonal matrix, we can use Theorem 1.1 to get the following recursive formulas:

$$\begin{aligned} q_1(\alpha) &= \frac{-\alpha}{2} \\ &= \frac{\alpha^2}{2} - 1 \\ &= -\alpha q_{n-1}(\alpha) - q_{n-2}(\alpha). \end{aligned}$$

Using the transformation $\alpha = -2x$, the family of the above polynomial can be transformed to a new family $\{T_n(x),\ n \geq 1\}$ where,

$$\begin{aligned} T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned}$$

Once again we observe that the family $\{T_n(x),\ n \geq 1\}$ is the set of Chebyshev polynomials of first kind. It is well known that for $x = \cos\theta$ the Chebyshev polynomials of the first kind can be written as

$$T_n(x) = \cos n\theta$$

which when equal to zero gives,

$$\theta_k = \frac{\pi(2k-1)}{2n}, \quad k = 1, 2, \cdots, n.$$

Therefore,

$$\begin{aligned} x_k &= \cos\theta_k \\ &= \cos\frac{\pi(2k-1)}{2n}, \quad k = 1, 2, \cdots, n. \end{aligned}$$

Applying the transformation $\alpha = -2x$, the eigenvalues of $S_n - ie_1e_2^T$ is given by

$$\alpha_k = -2\cos\frac{\pi(2k-1)}{2n}, \quad k = 1, 2, \cdots, n. \tag{3.4}$$

Thus, from (3.1),(3.3)and (3.4), we have

$$C_n = \frac{1}{2}[\prod_{1\leq k\leq n}(6 - 2\cos\frac{(2k-1)\pi}{2n})]$$

which is identical to the factorization(1.4).

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Prasanta Kumar Ray
C.V. Raman College of Engineering
Bhubaneswar -752054, India
E-mail address: rayprasanta2008@gmail.com