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Some New Properties of b-closed spaces

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ABSTRACT: In [5], the authors introduced the notion of b-closed spaces and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces.

Key Words: Topological spaces, b-open sets, b-θ-open sets.

Contents

1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively. A subset A of a topological space (X, τ) is called a b-open [1] (= γ-open [4]) set if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a b-open set is called a b-closed set. The intersection of all b-closed sets of X containing A is called the b-closure [1] of A and is denoted by $b\,\mathrm{Cl}(A)$. For each $x \in X$, the family of all b-open sets of (X, τ) containing a point x is denoted by $BO(X, x)$. The b-interior of A is the union of all b-open sets contained in A and is denoted by $b\,\mathrm{Int}(A)$. A set A is called a b-regular set [5] if it is both b-open and b-closed. The b-θ-closure [5]of a subset A, denoted by $b\,\mathrm{Cl}_\theta(A)$, is the set of all $x \in X$ such that $b\,\mathrm{Cl}(U) \cap A \neq \varnothing$ for every $U \in BO(X, x)$. A subset A is called b-θ-closed [5] if $A = b\,\mathrm{Cl}_\theta(A)$. By [5], it is proved that, for a subset A, $b\,\mathrm{Cl}_\theta(A)$ is the intersection of all b-θ-closed sets containing A. The complement of a b-θ-closed set is called a b-θ-open set. In [5], the authors introduced the notion of b-closed spaces and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces.

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2. $b(\theta)$-convergence and $b(\theta)$-adherence

Definition 2.1 *[2] A grill $\mathcal{G}$ on a topological space X is defined to be a collection of nonempty subsets of X such that (i) $A \in \mathcal{G}$ and $A \subset B \subset X \Rightarrow B \in \mathcal{G}$ and (ii) $A, B \subset X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

Definition 2.2 *A grill $\mathcal{G}$ on a topological space X is said to be:*

(i) $b(\theta)$-adhere at $x \in X$ if for each $U \in BO(X, x)$ and $G \in \mathcal{G}$, $b\,\mathrm{Cl}\,U \cap G \neq \emptyset$.

(ii) $b(\theta)$-converge to a point $x \in X$ if for each $U \in BO(X, x)$, there is some $G \in \mathcal{G}$, such that $G \subseteq b\,\mathrm{Cl}(U)2$.

Remark 2.3 *A grill $\mathcal{G}$ is $b(\theta)$-convergent to a point $x \in X$ if and only if $\mathcal{G}$ contains the collection $\{b\,\mathrm{Cl}(U) : U \in BO(X, x)\}$.*

Definition 2.4 *A filter $\mathcal{F}$ on a topological space X is said to $b(\theta)$-adhere at $x \in X$ ($b(\theta)$-converge to $x \in X$) if for each $F \in \mathcal{F}$ and each $U \in BO(X, x)$, $F \cap b\,\mathrm{Cl}(U) \neq \emptyset$ (resp. to each $U \in BO(X, x)$, there corresponds $F \in \mathcal{F}$ such that $F \subseteq b\,\mathrm{Cl}(U)$).*

Definition 2.5 *[6] If $\mathcal{G}$ is a grill (or a filter) on a topological space X, then the section of $\mathcal{G}$, denoted by $sec\mathcal{G}$, is given by,*
$sec\mathcal{G} = \{A \subseteq X : A \cap G \neq \emptyset$ for all $G \in \mathcal{G}\}$.

Theorem 2.6 *[6] Let X be a topological space. Then we have*

(i) For any grill (filter) $\mathcal{G}$ on X, $sec\mathcal{G}$ is a filter (resp. grill) on X.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are respectively a filter and a grill on X with $\mathcal{F} \subseteq \mathcal{G}$, then there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{G}$.

Theorem 2.7 *If a grill $\mathcal{G}$ on a topological space X, $b(\theta)$-adheres at some point $x \in X$, then $\mathcal{G}$ is $b(\theta)$-converges to x.*

Proof: Let a grill $\mathcal{G}$ on X, $b(\theta)$-adheres at some point $x \in X$. Then for each $U \in BO(X, x)$ and each $G \in \mathcal{G}$, $b\,\mathrm{Cl}(U) \cap G \neq \emptyset$ so that $b\,\mathrm{Cl}(U) \in sec\mathcal{G}$ for each $U \in BO(X, x)$, and hence $X \backslash b\,\mathrm{Cl}(U) \notin \mathcal{G}$. Then $b\,\mathrm{Cl}(U) \in \mathcal{G}$(as $\mathcal{G}$ is a grill and $X \in \mathcal{G}$) for each $U \in BO(X)$. Hence $\mathcal{G}$ must $b(\theta)$-converge to x. □

Example 2.8 *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then the grill $\mathcal{G}$ is $b(\theta)$-convergent but not $b(\theta)$-adheres.*

Definition 2.9 *Let X be a topological space. Then for any $x \in X$, we adopt the following notation:*
$\mathcal{G}(b(\theta), x) = \{A \subseteq X : x \in b\,\mathrm{Cl}_\theta(A)\}$,
$sec\mathcal{G}(b(\theta), x) = \{A \subseteq X : A \cap G \neq \emptyset$, for all $G \in \mathcal{G}(b(\theta), x)\}$.

Theorem 2.10 *A grill $\mathcal{G}$ on a topological space X, $b(\theta)$-adheres to a point x of X if and only if $\mathcal{G} \subseteq sec\mathcal{G}(b(\theta), x)$.*

Proof: A grill $\mathcal{G}$ on a topological space X, $b(\theta)$-adheres to a point x of X, we have $b\,\mathrm{Cl}(U) \cap G \neq \emptyset$ for all $U \in BO(X,x)$ and all $G \in \mathcal{G}$; hence $b\,\mathrm{Cl}_\theta(G)$ for all $G \in \mathcal{G}$. Then $G \in \mathcal{G}(b(\theta),x)$, for all $G \in \mathcal{G}$; hence $\mathcal{G} \subseteq \mathcal{G}(b(\theta),x)$. Conversely, let $\mathcal{G} \subseteq \mathcal{G}(b(\theta),x)$. Then for all $G \in \mathcal{G}$, $b\,\mathrm{Cl}(U) \cap G \neq \emptyset$, so that for all $U \in BO(X,x)$ and for all $G \in \mathcal{G}$, $b\,\mathrm{Cl}(U) \cap G \neq \emptyset$. Hence $\mathcal{G}$ $b(\theta)$-adheres at x. □

Theorem 2.11 *A grill $\mathcal{G}$ on a topological space X, $b(\theta)$-convergent to a point x of X if and only if sec$\mathcal{G}(b(\theta),x) \subseteq \mathcal{G}$.*

Proof: Let $\mathcal{G}$ be a grill on a topological space X, $b(\theta)$-convergent to a point $x \in X$. Then for each $U \in BO(X,x)$ there exists $G \in \mathcal{G}$ such that $G \subseteq b\,\mathrm{Cl}(U)$, and hence $b\,\mathrm{Cl}(U) \in \mathcal{G}$ for each $U \in BO(X,x)$. Now, $B \in \mathrm{sec}\mathcal{G}(b(\theta),x) \Rightarrow X\backslash B \notin \mathcal{G}(b(\theta),x) \Rightarrow x \notin b\,\mathrm{Cl}_\theta(X\backslash B) \Rightarrow$ there exists $U \in BO(X,x)$ such that $b\,\mathrm{Cl}(U) \cap (X\backslash B) = \emptyset \Rightarrow b\,\mathrm{Cl}(U) \subseteq B$, where $U \in BO(X,x) \Rightarrow B \in \mathcal{G}$. Conversely, let if possible, $\mathcal{G}$ not to $b(\theta)$-converge to x. Then for some $U \in BO(X,x)$, $b\,\mathrm{Cl}(U) \notin \mathcal{G}$ and hence $b\,\mathrm{Cl}(U) \notin \mathrm{sec}\mathcal{G}(b(\theta),x)$. Thus for some $A \in \mathcal{G}(b(\theta),x)$, $A \cap b\,\mathrm{Cl}(U) = \emptyset$. But $A \in \mathcal{G}(b(\theta),x) \Rightarrow x \in b\,\mathrm{Cl}_\theta(A) \Rightarrow b\,\mathrm{Cl}(A) \cap U \neq \emptyset$. □

3. b-closedness and grills

Definition 3.1 *A nonempty subset A of a topological space X is called b-closed relative to X [5] if for every cover $\mathcal{U}$ of A by b-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\{b\,\mathrm{Cl}\,U : U \in \mathcal{U}_0\}$. If, in addition, $A = X$, then X is called a b-closed space.*

Theorem 3.2 *For a topological space X, the following statements are equivalent:*

(i) X is b-closed;

(ii) Every maximal filter base $b(\theta)$-converges to some point of X;

(iii) Every filter base $b(\theta)$-adhere to some point of X;

(iv) For every family $\{V_\alpha : \alpha \in I\}$ of b-closed sets that $\cap\{V_i : i \in I\} = \emptyset$, there exists a finite subset I_0 of I such that $\bigcap\{b\,\mathrm{Int}(V_i) : i \in I_0\} = \emptyset$.

Proof: $(i) \Rightarrow (ii)$: Let $\mathcal{F}$ be a maximal filter base on X. Suppose that $\mathcal{F}$ does not b-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not b-θ-accumulate at any point of X. For each $x \in X$, there exist $F_x \in \mathcal{F}$ and $V_x \in BO(X,x)$ such that $b\,\mathrm{Cl}(V_x) \cap F_x = \emptyset$. The family $\{V_x : x \in X\}$ is a cover of X by b-open sets of X. By (i), there exists a finite number of points $x_1, x_2, ..., x_n$ of X such that $X = \cup\{b\,\mathrm{Cl}(V_{x_i}) : i = 1,2,...,n\}$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap\{F_{x_i} : i = 1,2,...,n\}$. Therefore, we obtain $F_0 = \emptyset$. This is a contradiction. $(ii) \Rightarrow (iii)$: Let $\mathcal{F}$ be any filter base on X. Then, there exists a maximal filter base $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ b-θ-converges to some point $x \in X$. For every $F \in \mathcal{F}$ and every $V \in BO(X,x)$, there exists $F_0 \in \mathcal{F}_0$

such that $F_0 \subseteq b\,\mathrm{Cl}(V)$; hence $\emptyset \neq F_0 \cap F \subseteq b\,\mathrm{Cl}(V) \cap F$. This shows that $\mathcal{F}$ b-θ-accumulates at x. $(iii) \Rightarrow (iv)$: Let $\{V_\alpha : \alpha \in I\}$ be any family of b-closed subsets of X such that $\cap\{V_\alpha : \alpha \in I\} = \emptyset$. Let $\Gamma(I)$ denote the ideal of all finite subsets of A. Assume that $\cap\{b\,\mathrm{Int}(V_\alpha) : \alpha \in I\} = \emptyset$ for every $I \in \Gamma(I)$. Then, the family $\mathcal{F} = \{\cap_{\alpha \in I} b\,\mathrm{Int}(V_\alpha) : I \in \Gamma(I)\}$ is a filter base on X. By (iii), $\mathcal{F}$ b-θ-accumulates at some point $x \in X$. Since $\{X \backslash V_\alpha : \alpha \in I\}$ is a cover of X, $x \in X \backslash V_{\alpha_0}$ for some $\alpha_0 \in I$. Therefore, we obtain $X \backslash V_{\alpha_0} \in BO(X, x)$, $b\,\mathrm{Int}(V_{\alpha_0}) \in \mathcal{F}$ and $b\,\mathrm{Cl}(X \backslash V_{\alpha_0}) \cap b\,\mathrm{Int}(V_{\alpha_0}) = \emptyset$, which is a contradiction. $(iv) \Rightarrow (i)$: Let $\{V_\alpha : \alpha \in I\}$ be a cover of X by b-open sets of X. Then $\{X \backslash V_\alpha : \alpha \in I\}$ is a family of b-closed subsets of X such that $\cap\{X \backslash V_\alpha : \alpha I\} = \emptyset$. By (iv), there exists a finite subset I_0 of I such that $\cap\{b\,\mathrm{Int}(X \backslash V_\alpha) : \alpha \in I_0\} = \emptyset$; hence $X = \cup\{b\,\mathrm{Cl}(V_\alpha) : \alpha \in I_0\}$. This shows that X is b-closed. □

Theorem 3.3 *A topological space X is b-closed if and only if every grill on X is $b(\theta)$-convergent in X.*

Proof: Let $\mathcal{G}$ be any grill on a b-closed space X. Then by Theorem 2.6, sec$\mathcal{G}$ is a filter on X. Let $B \in$ sec$\mathcal{G}$, then $X \backslash B \notin \mathcal{G}$ and hence $B \in \mathcal{G}$(as $\mathcal{G}$ is a grill). Thus sec$\mathcal{G} \subseteq \mathcal{G}$. Then by Theorem 2.6(ii), there exists an ultrafilter $\mathcal{U}$ on X such that sec$\mathcal{G} \subseteq \mathcal{U} \subseteq \mathcal{G}$. Now as X is b-closed, in view of Theorem 3.2, the ultrafilter $\mathcal{U}$ is $b(\theta)$-convergent to some point $x \in X$. Then for each $U \in BO(X, x)$, there exists $F \in \mathcal{U}$ such that $F \subseteq b\,\mathrm{Cl}(U)$. Consequently, $b\,\mathrm{Cl}(U) \in \mathcal{U} \subseteq \mathcal{G}$, That is $b\,\mathrm{Cl}(U) \in \mathcal{G}$, for each $U \in BO(X, x)$. Hence $\mathcal{G}$ is $b(\theta)$-convergent to x. Conversely, let every grill on X be $b(\theta)$-convergent to some point of X. By virtue of Theorem 3.2 it is enough to show that every ultrafilter on X is $b(\theta)$-converges in X, which is immediate from the fact that an ultrafilter on X is also a grill on X. □

Theorem 3.4 *A topological space X is b-closed relative to X if and only if every grill $\mathcal{G}$ on X with $A \in \mathcal{G}$, $b(\theta)$-converges to a point in A.*

Proof: Let A be b-closed relative to X and $\mathcal{G}$ a grill on X satisfying $A \in \mathcal{G}$ such that $\mathcal{G}$ does not $b(\theta)$-converge to any $a \in A$. Then to each $a \in A$, there corresponds some $U_a \in BO(X, a)$ such that $b\,\mathrm{Cl}(U_a) \notin \mathcal{G}$. Now $\{U_a : a \in A\}$ is a cover of A by b-open sets of X. Then $A \subseteq \bigcup_{i=1}^{n} b\,\mathrm{Cl}(U_{a_i}) = U$ (say) for some positive integer n. Since $\mathcal{G}$ is a grill, $U \notin \mathcal{G}$; hence $A \notin \mathcal{G}$, which is a contradiction. Conversely, let A be not b-closed relative to X. Then for some cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of A by b-open sets of X, $\mathcal{F} = \{A \backslash \cup_{\alpha \in I_0} b\,\mathrm{Cl}(U_\alpha) : I_0$ is finite subset of $I\}$ is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a grill on X with $A \in \mathcal{F}^*$ (as each F of $\mathcal{F}$ is a subset of A). Now for each $x \in A$, there must exists $\beta \in I$ such that $x \in U_\beta$, as $\mathcal{U}$ is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A \backslash b\,\mathrm{Cl}(U_\beta)) \neq \emptyset$, so that $G \supset b\,\mathrm{Cl}(U_\beta)$ for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot $b(\theta)$-converge to any point of A. The contradiction proves the desired result. □

Theorem 3.5 *If X is any topological space such that every grill $\mathcal{G}$ on X with the property that $\bigcap_{i=1}^{n} b\operatorname{Cl}_\theta(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ..., G_n\}$ of $\mathcal{G}$, $b(\theta)$-adheres in X, then X is a b-closed space.*

Proof: Let $\mathcal{U}$ be an ultrafilter on X. Then $\mathcal{U}$ is a grill on X and also for each finite subcollection $\{U_1, U_2, ..., U_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} b\operatorname{Cl}_\theta(U_i) \supseteq \bigcap_{i=1}^{n} U_i \neq \emptyset$, so that $\mathcal{U}$ is a grill on X with the given condition. Hence by hypothesis, $\mathcal{U}$, $b(\theta)$-adheres. Consequently, by Theorem 3.2, X is b-closed. □

Theorem 3.6 *[5] For any $A \subseteq X$, $b\operatorname{Cl}_\theta(A) = \cap\{b\operatorname{Cl} U : A \subseteq U \in BO(X)\}$.*

Definition 3.7 *A grill $\mathcal{G}$ on a topological space X is said to be:*
(a) $b(\theta)$-linked if for any two members $A, B \in \mathcal{G}$, $b\operatorname{Cl}_\theta(A) \cap b\operatorname{Cl}_\theta(B) \neq \emptyset$,
(b) $b(\theta)$-conjoint if for every finite subfamily $A_1, A_2, ..., A_n$ of $\mathcal{G}$,
$b\operatorname{Int}(\bigcap_{i=1}^{n} b\operatorname{Cl}_\theta(A_i)) \neq \emptyset$.

It is clear that every $b(\theta)$-conjoint grill is $b(\theta)$-linked. The following example shows that the converse is need not be true in general.

Example 3.8 *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \{\{c\}, \{b, c\}, \{a, c\}, X\}$. Then the grill $\mathcal{G}$ is $b(\theta)$-linked but not $b(\theta)$-conjoint.*

Theorem 3.9 *In a b-closed space X, every $b(\theta)$-conjoint grill $b(\theta)$-adheres in X.*

Proof: Consider any $b(\theta)$-conjoint grill $\mathcal{G}$ on a b-closed space X. We first note from Theorem 3.5 that for $A \subseteq X$, $b\operatorname{Cl}_\theta(A)$ is b-closed (as an arbitrary intersection of b-closed sets is b-closed). Thus $\{b\operatorname{Cl}_\theta(A) : A \in \mathcal{G}\}$ is a collection of b-closed sets in X such that $b\operatorname{Int}(\bigcap_{i=1}^{n} b\operatorname{Cl}_\theta(A_i)) \neq \emptyset$ for any finite subcollection $A_1, A_2, ..., A_n$ of $\mathcal{G}$. Then $b\operatorname{Int}(\bigcap_{i=1}^{n}(b\operatorname{Cl}_\theta(A_i))) \neq \emptyset$ for any finite subcollection $A_1, A_2, ..., A_n$ of $\mathcal{G}$. Thus by Theorem 3.2, $\cap_{\alpha \in I}\{b\operatorname{Cl}_\theta(A) : A \in \mathcal{G}\} \neq \emptyset$, That is there exists $x \in X$ such that $x \in b\operatorname{Cl}_\theta(A)$ for all $A \in \mathcal{G}$. Hence $\mathcal{G} \subseteq \mathcal{G}(b(\theta), x)$ so that by Theorem 2.10, $\mathcal{G}$, $b(\theta)$-adheres at $x \in X$. □

Definition 3.10 *A subset A of a topological space X is called b-regular open if $A = b\operatorname{Int}(b\operatorname{Cl}(A))$. The complement a b-regular open set is called a b-regular closed set.*

Definition 3.11 *A topological space X is called b-almost regular if for each $x \in X$ and each b-regular open set V in X with $x \in V$, there is a b-regular open set U in X such that $x \in U \subseteq b\operatorname{Cl}(U) \subseteq V$.*

Theorem 3.12 *In a b-almost regular b-closed space X, every grill $\mathcal{G}$ on X with the property $\bigcap_{i=1}^{n} b\operatorname{Cl}_\theta(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ..., G_n\}$ of $\mathcal{G}$, $b(\theta)$-adheres in X.*

Proof: Let X be a b-almost regular b-closed space and $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ a grill on X with the property that $\cap_{\alpha\in I_0} b\,\mathrm{Cl}_\theta(G_\alpha) \neq \emptyset$ for every finite subset I_0 of I. We consider $\mathcal{F} = \{\cap_{\alpha\in I_0} b\,\mathrm{Cl}_\theta\, G_\alpha : I_0$ is a finite subfamily of $I\}$. Then $\mathcal{F}$ is a filterbase on X. By the b-closedness of X, $\mathcal{F}$, $b(\theta)$-adheres at some $x \in X$, that is, $x \in b\,\mathrm{Cl}_\theta(b\,\mathrm{Cl}_\theta(G))$ for all $G \in \mathcal{G}$, that is, $\mathcal{G} \subseteq (b(\theta), x)$. Hence by Theorem 2.10, $\mathcal{G}$ $b(\theta)$-adheres at $x \in X$. □

Corollary 3.13 *In a b-almost regular space X, the following statements are equivalent:*

(i) Every grill $\mathcal{G}$ on X with the property that $\bigcap_{i=1}^{n} b\,\mathrm{Cl}_\theta(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ..., G_n\}$ of $\mathcal{G}$, $b(\theta)$-adheres in X.

(ii) X is b-closed.

(iii) Every $b(\theta)$-conjugate grill $b(\theta)$-adheres in X.

Theorem 3.14 *Every grill $\mathcal{G}$ on a topological space X with the property that $\cap\{b\,\mathrm{Cl}_\theta(G) : G \in \mathcal{G}_0\} \neq \emptyset$ for every finite subsets $\mathcal{G}_0$ of $\mathcal{G}$, $b(\theta)$-adheres in X if and only if for every family $\mathcal{F}$ of subsets of X for which the family $\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property, we have $\cap\{b\,\mathrm{Cl}_\theta(F) : \mathcal{F}\} \neq \emptyset$.*

Proof: Let every grill on a topological space X satisfying the given condition, $b(\theta)$-adhere in X, and suppose that $\mathcal{F}$ is a family of subsets of X such that the family $\mathcal{F}^* = \{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property. Let $\mathcal{U}$ be the collection of all those families $\mathcal{G}$ of subsets of X for which $\mathcal{G}^* = \{b\,\mathrm{Cl}_\theta(G) : \mathcal{G}\}$ has the finite intersection property and $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F} \in \mathcal{U}$ is a partially ordered set under set inclusion in which every chain clearly has an upper bound. By Zorn's lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a grill with the stipulated property. Hence $\cap\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\} \supseteq \cap\{b\,\mathrm{Cl}_\theta(U) : F \in \mathcal{U}^*\} \neq \emptyset$. Conversely, if $\mathcal{F}$ is a grill on X with the given property, then for every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\cap\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F} \neq \emptyset\}$. So, by hypothesis, $\cap\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\} \neq \emptyset$. Hence $\mathcal{F}$, $b(\theta)$-adheres in X. □

Definition 3.15 *A topological space X is called $b(\theta)$-linkage b-closed if every $b(\theta)$-linked grill on X, $b(\theta)$-adheres.*

Theorem 3.16 *Every $b(\theta)$-linkage b-closed space is b-closed.*

Proof: The proof is clear. □

Proposition 3.17 *[5] Let A be a subset of a topological space (X, τ). Then:*

(i) If $A \in BO(X)$, then $b\,\mathrm{Cl}(A) = b\,\mathrm{Cl}_\theta(A)$.

(ii) If A is b-regular, then A is b-θ-closed.

Theorem 3.18 *In the class of b-almost regular spaces, the concept of b-closedness and $b(\theta)$-linkage b-closedness become identical.*

Proof: In view of Theorem 3.16, it is enough to show that a b-almost regular b-closed space is $b(\theta)$-linkage b-closed. Let $\mathcal{G}$ be any $b(\theta)$-linked grill on a b-almost regular b-closed space X such that $\mathcal{G}$ does not $b(\theta)$-adhere in X. Then for each $x \in X$, there exists $G_x \in \mathcal{G}$ such that $x \notin b\,\mathrm{Cl}_\theta(G_x) = b\,\mathrm{Cl}_\theta(b\,\mathrm{Cl}_\theta(G_x))$. Then there exists $U_x \in BO(X, x)$ such that $b\,\mathrm{Cl}(U_x) \cap b\,\mathrm{Cl}_\theta(G_x) = \emptyset$, which gives $b\,\mathrm{Cl}_\theta(U_x) \cap b\,\mathrm{Cl}_\theta(G_x) = \emptyset$ by Proposition 3.17, $b\,\mathrm{Cl}_\theta(U) = b\,\mathrm{Cl}(U))$. Since $b\,\mathrm{Cl}_\theta(G_x) \in \mathcal{G}$ and $\mathcal{G}$ is a $b(\theta)$-linked grill on X, $b\,\mathrm{Cl}_\theta(U_x) = b\,\mathrm{Cl}(U_x) \notin \mathcal{G}$. Now, $\{U_x : x \in X\}$ is a cover of X by b-open sets of X. So by b-closedness of X, $X = \cup\{b\,\mathrm{Cl}(U_{x_i}) : i = 1, 2, ..., n\}$, for a finite subset $\{x_1, x_2, ..., x_n\}$ of X. It is then follows that $x \notin \mathcal{G}$ for $i = 1, 2, ..., n$, which is a contradiction. Hence $\mathcal{G}$ must $b(\theta)$-adhere in X, proving X to be $b(\theta)$-linkage b-closed. □

Definition 3.19 *A topological space X is said to be b-compact [3] if every cover $\mathcal{U}$ of X by b-open sets of X has a finite subcover.*

Definition 3.20 *A topological space X is $b(\theta)$-regular if every grill on X which $b(\theta)$-converges must b-converge (not necessarily to the same point), where b-convergence of a grill is defined in the usual way. That is a grill $\mathcal{G}$ on X is said to b-converge to $x \in X$ if $BO(X, x) \subseteq \mathcal{G}$.*

Theorem 3.21 *A topological space X is b-compact if and only if every grill b-converges.*

Proof: Let $\mathcal{G}$ be a grill on a b-compact space such that $\mathcal{G}$ does not b-converge to any point $x \in X$. Then for each $x \in X$, there exists $U_x \in BO(X, x)$ with $(*)$ $U_x \notin \mathcal{G}$. As $\{U_x : x \in X\}$ is a cover of the b-compact space X by b-open sets, there exist finitely many points $x_1, x_2, ..., x_n$ in X such that $X = \bigcap_{i=1}^{n} U_{x_i}$. Since $X \in \mathcal{G}$ for some i, $(1 \leq i \leq n)$, $U_{x_i} \in \mathcal{G}$, which goes against $(*)$. Conversely, let every grill on X b-converge and if possible, let X be not b-compact. Then there exists a cover $\mathcal{U}$ of X by b-open sets of X having no finite subcover. Then $\mathcal{F} = \{X \backslash \cup \mathcal{U}_0 : \mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ is a filterbase on X. Then $\mathcal{F}$ is contained in an ultrafilter $\mathcal{G}$, and then $\mathcal{G}$ b-converges to some point x of X. Then for some $U \in \mathcal{U}$, $x \in U$, and hence $U \in \mathcal{G}$. But $X \backslash U \in \mathcal{F} \subseteq \mathcal{U}$. Thus U and $X \backslash U$ both belong to $\mathcal{U}$, which is a filter, so giving a contradiction. □

Theorem 3.22 *A b-compact space X is b-closed, while the converse is also true if X is $b(\theta)$-regular.*

Proof: The proof is clear. □

Definition 3.23 *A topological space (X, τ) is said to be b-regular [5] if for any closed set $F \subset X$ and any point $x \in X \backslash F$, there exists disjoint b-open sets U and V such that $x \in U$ and $F \subset V$.*

Theorem 3.24 *A topological space X is b-regular [5] if and only if for each $x \in X$ and each $U \in BO(X, x)$, there exists $V \in BO(X, x)$ such that $b\,\mathrm{Cl}(V) \subseteq U$.*

Theorem 3.25 *Every b-regular space is $b(\theta)$-regular.*

Proof: Let $\mathcal{G}$ be a grill on a b-regular X, $b(\theta)$-converging to a point x of X. For each $U \in BO(X, x)$, there exists, by b-regularity of X, a $V \in BO(X, x)$ such that $b\,\mathrm{Cl}(V) \subseteq U$. By hypothesis, $b\,\mathrm{Cl}(V) \in \mathcal{G}$. Hence $\mathcal{G}$ b-converges to x, proving X to be $b(\theta)$-regular. □

Example 3.26 *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Clearly, X is b-compact. Hence by Theorem 3.21, every grill on X must b-converge in X. Thus, X is $b(\theta)$-regular. But it is easy to check that X is not b-regular.*

Theorem 3.27 *If a topological space X is b-closed b-regular, then X is b-compact.*

Proof: Let X be a b-closed and b-regular space. Let $\{V_\alpha : \alpha \in I\}$ be any open cover of X. For each $x \in X$, there exists an $\alpha(x) \in I$ such that $x \in V_{\alpha(x)}$. Since X is b-regular, there exists $U(x) \in BO(X, x)$ such that $U(x) \subset b\,\mathrm{Cl}(U(x)) \subset V_{\alpha(x)}$. Then, $\{U(x); x \in X\}$ is a b-open cover of the b-closed space X and hence there exists a finite amount of points, say, $x_1, x_2, \ldots.., x_n$ such that $X = \bigcup_{i=1}^{n} b\,\mathrm{Cl}(U(x_i)) = \bigcup_{i=1}^{n} V_{\alpha(x_i)}$. This shows that X is compact. □

4. Sets which are b-closed relative to a space

Theorem 4.1 *For a topological space X, the following statements are equivalent:*

(i) A is b-closed relative to X;

(ii) Every maximal filter base $b(\theta)$-converges to some point of X;

(iii) Every filter base $b(\theta)$-adhere to some point of X;

(iv) For every family $\{V_\alpha : \alpha \in I\}$ of b-closed sets such that $\cap\{V_i : i \in I\} \cap A = \emptyset$, there exists a finite subset I_0 of I such that $\bigcap\{b\,\mathrm{Int}(V_i) : i \in I_0\} \cap A = \emptyset$.

Proof: The proof is clear. □

Theorem 4.2 *If X is a b-closed space, then every cover of X by b-θ-open set has a finite subcover.*

Proof: Let $\{V_\alpha : \alpha \in I\}$ be any cover of X by b-θ-open subsets of X. For each $x \in X$, there exists $\alpha(x) \in I$ such that $x \in V_{\alpha(x)}$ is b-θ-open, there exists $V_x \in BO(X,x)$ such that $V_x \subseteq b\,\mathrm{Cl}(V_x) \subseteq V_{\alpha(x)}$. The family $\{V_x : x \in X\}$ is a b-open cover of X. Since X is b-closed, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ such that $X = \overset{n}{\underset{i=1}{\cup}} b\,\mathrm{Cl}(V_{x_i})$. Therefore, we obtain that $X = \overset{n}{\underset{i=1}{\cup}} V_{x_i}$. □

Theorem 4.3 *Let A, B be subsets of a topological space X. If A is b-θ-closed and B is b-closed relative to X, then $A \cap B$ is b-closed relative to X.*

Proof: Let $\{V_\alpha : \alpha \in I\}$ be any cover of $A \cap B$ by b-open subsets of X. Since $X\backslash A$ is b-θ-open, for each $x \in B\backslash A$ there exists $W_x \in BO(X,x)$ such that $b\,\mathrm{Cl}(W_x) \subseteq X\backslash A$. The family $\{W_x : x \in B\backslash A\} \cup \{V_\alpha : \alpha \in I\}$ is a cover of B by b-open sets of X. Since B is b-closed relative to X, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in $B\backslash A$ and a finite subset I_0 of I such that $B \subseteq \overset{n}{\underset{i=1}{\cup}} b\,\mathrm{Cl}(W_{x_i}) \cup \underset{\alpha\in I_0}{\cup} b\,\mathrm{Cl}(V_\alpha)$. Since $b\,\mathrm{Cl}(W_{x_i}) \cap A = \emptyset$ for each i, we obtain that $A \cap B \subseteq \cup\{b\,\mathrm{Cl}(V_\alpha) : \alpha \in I_0\}$. This shows that $A \cap B$ is b-closed relative to X. □

Corollary 4.4 *If K is b-θ-closed of a b-closed space X, then K is b-closed relative to X.*

Definition 4.5 *A topological space X is called b-connected [5] if X cannot be expressed as the union of two disjoint b-open sets. Otherwise, X is b-disconnected.*

Theorem 4.6 *Let X be a b-disconnected space. Then X is b-closed if and only if every b-regular subset of X is b-closed relative to X.*

Proof: Necessity: Every b-regular set is b-θ-closed by Proposition 3.17. Since X is b-closed, the proof is completed by Corellary 4.4.
Sufficiency: Let $\{V_\alpha : \alpha \in I\}$ be any cover of X by b-open subsets of X. Since X is b-disconnected , there exists a proper b-regular subset A of X. By our hypothesis, A and $X\backslash A$ are b-closed relative to X. There exist finite subsets A_1 and A_2 of A such that $A \subseteq \underset{\alpha\in A_1}{\cup} b\,\mathrm{Cl}(V_\alpha)$, $X\backslash A \subseteq \underset{\alpha\in A_2}{\cup} b\,\mathrm{Cl}(V_\alpha)$. Therefore, we obtain that $X = \cup\{b\,\mathrm{Cl}(V_\alpha) : \alpha \in A_1 \cup A_2\}$. □

Theorem 4.7 *If there exists a proper b-regular subset A of a topological space X such that A and $X\backslash A$ are b-closed relative to X, then X is b-closed.*

Proof: This proof is similar to the Theorem 4.6 and hence omitted. □

Definition 4.8 *A function $f : (X,\tau) \to (Y,\sigma)$ is called b-irresolute [4] if $f^{-1}(V)$ is b-open in X for every b-open subset V of Y.*

Lemma 4.9 *[4] A function $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute if and only if for each subset A of X, $f(b\,\mathrm{Cl}(A)) \subseteq b\,\mathrm{Cl}(f(A))$.*

Theorem 4.10 *If a function $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute surjection and K is b-closed relative to X, then $f(K)$ is b-closed relative to Y.*

Proof: Let $\{V_\alpha : \alpha \in I\}$ be any cover of $f(K)$ by b-open subsets of Y. Since f is b-irresolute, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of K by b-open subsets of X, where K is b-closed relative to X. Therefore, there exists a finite subset I_0 of I such that $K \subseteq \bigcup_{\alpha \in A_0} b\,\mathrm{Cl}(f^{-1}(V_\alpha))$. Since f is b-irresolute surjective, by Lemma 4.9, we have $f(K) \subseteq \bigcup_{\alpha \in A_0} f(b\,\mathrm{Cl}(f^{-1}(V_\alpha)) \subseteq \bigcup_{\alpha \in A_0} f(b\,\mathrm{Cl}(V_\alpha)$. □

Corollary 4.11 *If a function $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute surjection and X is b-closed, then Y is b-closed.*

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