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On decompositions of $\mathcal{I}$-rg-continuity

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ABSTRACT: We introduce the notions of $\mathcal{I}$-rg-open sets, $\mathcal{I}$-$g\alpha^{**}$-open sets, $\mathcal{I}$-gpr-open sets, $\mathcal{I}$-C_r-sets and $\mathcal{I}$-${C_r}^*$-sets to obtain the decompositions of $\mathcal{I}$-rg-continuity in ideal topological spaces.

Key Words: $\mathcal{I}$-rg-open sets, $\mathcal{I}$-$g\alpha^{**}$-open sets, $\mathcal{I}$-gpr- open sets, $\mathcal{I}$-C_r-sets, $\mathcal{I}$-${C_r}^*$-sets and $\mathcal{I}$-rg-continuity

Contents

1. Introduction and Preliminaries

Acikgoz and Yuksel [1] introduced the concept of $\mathcal{I}$-R-closed sets and obtained new decompositions of some weaker forms of continuity. Recently in 2010, Noiri et.al [3] introduced the notions of $g\alpha^{**}$-$\mathcal{I}$-open sets, gpr- $\mathcal{I}$-open sets, C_r-$\mathcal{I}$-open sets and ${C_r}^*$-$\mathcal{I}$-open sets to obtain the decompositions of rg-continuity in ideal topological spaces. In this paper we introduce the notions of $\mathcal{I}$-regular-closed sets, $\mathcal{I}$-$g\alpha^{**}$-open sets, $\mathcal{I}$-gpr- open sets, $\mathcal{I}$-C_r-sets and $\mathcal{I}$-${C_r}^*$-sets to obtain the new decompositions of $\mathcal{I}$-rg-continuity in ideal topological spaces. An *ideal* $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a *local function* [6] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I},\tau)=\{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(X)\}$. It is well known that $cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology τ^* finer than τ [2] and $int^*(A)$ will denote the interior of A in $(X,\tau^*,\mathcal{I})$. When there is no chance of confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\tau,\mathcal{I})$. If $\mathcal{I}$ is an ideal on X then $(X,\tau,\mathcal{I})$ is called an ideal topological space or an ideal space. Throughout this paper X denotes the ideal topological space $(X,\tau,\mathcal{I})$. A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be an $\mathcal{I}$-pre-open [4](resp. $\mathcal{I}$-α-open [4]) if $A \subseteq int^*(cl^*(A))$ (resp. $A \subseteq int^*(cl^*(int^*(A))))$. For a subset A of an ideal topological space $(X,\tau,\mathcal{I})$, the

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$\mathcal{I}$-pre-interior (resp. $\mathcal{I}$-α-interior) of A, denoted by $\mathcal{I}$-pint(A) [4](resp. $\mathcal{I}$-αint(A) [4]) is defined as the union of all $\mathcal{I}$-pre-open (resp. $\mathcal{I}$-α-open) sets of $(X, \tau, \mathcal{I})$, contained in A. A subset A of $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}$-t-set(resp. $\mathcal{I}$-α^*-set) [5] if $int^*(A)$=$int^*(cl^*(A))$(resp. $int^*(A)$= $int^*(cl^*(int^*(A)))$.

Lemma 1.1 *[4] For a subset A of $(X, \tau, \mathcal{I})$, we have*

1. *$\mathcal{I}$-pint(A) $= A \cap int^*(cl^*(A))$*
2. *$\mathcal{I}$-αint(A) $= A \cap int^*(cl^*(int^*(A)))$*

2. $\mathcal{I}$-rg-open sets, $\mathcal{I}$-$g\alpha^{**}$-open sets and $\mathcal{I}$-gpr-open sets

Definition 2.1 *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-regular closed if $A = cl^*(int^*(A))$.*

The complement of $\mathcal{I}$-regular closed set is $\mathcal{I}$-regular open.

Definition 2.2 *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

1. *$\mathcal{I}$-rg-open if $F \subseteq int^*(A)$ whenever $F \subseteq A$ and F is $\mathcal{I}$-regular closed.*
2. *$\mathcal{I}$-$g\alpha^{**}$-open if $F \subseteq \mathcal{I}$-$\alpha int(A)$ whenever $F \subseteq A$ and F is $\mathcal{I}$-regular closed.*
3. *$\mathcal{I}$-gpr-open if $F \subseteq \mathcal{I}$-$pint(A)$ whenever $F \subseteq A$ and F is $\mathcal{I}$-regular closed.*

Proposition 2.3 *For a subset of an ideal topological space $(X, \tau, \mathcal{I})$ the following hold:*

1. *An $\mathcal{I}$-rg-open set is $\mathcal{I}$-$g\alpha^{**}$-open.*
2. *An $\mathcal{I}$-$g\alpha^{**}$-open set is $\mathcal{I}$-gpr-open.*
3. *An $\mathcal{I}$-rg-open set is $\mathcal{I}$-gpr-open.*

Proof:

1. Let A be an $\mathcal{I}$-rg-open set. Then for any $\mathcal{I}$-regular closed F with $F \subseteq A$, we have $F \subseteq int^*(A) \subseteq (int^*(int^*(A))^*) \cup int^*(A)$= $int^*(int^*(A))^* \cup int^*(int^*(A)) \subseteq int^*((int^*(A))^* \cup int^*(A))$= $int^*(cl^*(int^*(A)))$. So, $F \subseteq A \cap int^*(cl^*(int^*(A)))$=$\mathcal{I}$-$\alpha int(A)$. Hence A is $\mathcal{I}$-$g\alpha^{**}$-open.
2. Let A be an $\mathcal{I}$-$g\alpha^{**}$-open set and F be any $\mathcal{I}$-regular closed set with $F \subseteq A$. Then we have $F \subseteq \mathcal{I}$-$\alpha int(A)$= $A \cap int^*(cl^*(int^*(A))) \subseteq A \cap int^*(cl^*(A))$=$\mathcal{I}$-pint(A), which implies that A is $\mathcal{I}$-gpr-open.
3. Proof is similar to the proofs of 1 and 2.

□

Converses need not be true as seen from the following examples.

Example 2.4 *Let* $X = \{a, b, c, d, e, f\}$, $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, c, d\}, X\}$ *and* $\mathcal{I} = \{\phi, \{d\}\}$. *Then the set* $\{a, c, d, e, f\}$ *is* $\mathcal{I}$-$g\alpha^{**}$-*open but not* $\mathcal{I}$-rg-*open.*

Example 2.5 *Let* $X = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$ *and* $\mathcal{I} = \{\phi, \{d\}\}$. *Then the set* $\{a, c, d, e\}$ *is* $\mathcal{I}$-gpr-*open but neither* $\mathcal{I}$-rg-*open nor* $\mathcal{I}$-$g\alpha^{**}$-*open.*

3. $\mathcal{I}$-C_r-sets and $\mathcal{I}$-$C_r{}^*$-sets

Definition 3.1 *A subset* A *of an ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be*

1. $\mathcal{I}$-C_r-*set if* $A = U \cap V$ *where* U *is* $\mathcal{I}$-rg-*open and* V *is an* $\mathcal{I}$-t-*set.*
2. $\mathcal{I}$-$C_r{}^*$-*set if* $A = U \cap V$ *where* U *is* $\mathcal{I}$-rg-*open and* V *is an* $\mathcal{I}$-α^*-*set.*

Proposition 3.2 *For a subset of an ideal topological space, the following properties hold:*

1. *An* $\mathcal{I}$-t-*set is an* $\mathcal{I}$-α^*-*set and an* $\mathcal{I}$-C_r-*set.*
2. *An* $\mathcal{I}$-α^*-*set is an* $\mathcal{I}$-$C_r{}^*$-*set.*
3. *An* $\mathcal{I}$-C_r-*set is an* $\mathcal{I}$-$C_r{}^*$-*set.*
4. *An* $\mathcal{I}$-rg-*open set is an* $\mathcal{I}$-C_r-*set and an* $\mathcal{I}$-$C_r{}^*$-*set.*

Remark 3.3 *From Proposition 3.2, we have the following diagram in which none of the implications is reversible.*

$$\begin{array}{ccccc} \mathcal{I}\text{-}rg\text{-open} & \longrightarrow & \mathcal{I}\text{-}C_r\text{-set} & \longleftarrow & \mathcal{I}\text{-}t\text{-set} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{I}\text{-}C_r^*\text{-set} & \longleftarrow & \mathcal{I}\text{-}\alpha^*\text{-set} \end{array}$$

Example 3.4 *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ *and* $\mathcal{I} = \{\phi, \{c\}\}$. *Then the set* $\{a, b\}$ *is an* $\mathcal{I}$-C_r-*set and an* $\mathcal{I}$-$C_r{}^*$-*set but neither an* $\mathcal{I}$-t-*set nor an* $\mathcal{I}$-α^*-*set.*

Example 3.5 *Let* $X = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$ *and* $\mathcal{I} = \{\phi, \{d\}\}$. *Then the set* $\{a, c, d, e\}$ *is an* $\mathcal{I}$-$C_r{}^*$-*set but not an* $\mathcal{I}$-C_r *set.*

Example 3.6 *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, X\}$ *and* $\mathcal{I} = \{\phi, \{c\}\}$. *Then the set* $\{a, b, c\}$ *is an* $\mathcal{I}$-C_r *set and an* $\mathcal{I}$-$C_r{}^*$ *set but not* $\mathcal{I}$-rg-*open.*

Theorem 3.7 *A subset* A *of* $(X, \tau, \mathcal{I})$ *is* $\mathcal{I}$-rg-*open if and only if it is both* $\mathcal{I}$-gpr-*open and an* $\mathcal{I}$-C_r-*set in* $(X, \tau, \mathcal{I})$.

Proof: Necessity: Assume that A is $\mathcal{I}$-rg-open in $(X, \tau, \mathcal{I})$. By Propositions 2.3 and 3.2, we have A is both $\mathcal{I}$-*gpr*-open and an $\mathcal{I}$-C_r-set in $(X, \tau, \mathcal{I})$.
Sufficiency: Assume that A is $\mathcal{I}$-*gpr*-open and an $\mathcal{I}$-C_r-set in $(X, \tau, \mathcal{I})$. Let $F \subseteq A$ and F is $\mathcal{I}$-regular closed. Since A is an $\mathcal{I}$-C_r-set, $A = U \cap V$, where U is $\mathcal{I}$-rg-open

and V is an $\mathcal{I}$-t-set. Since A is $\mathcal{I}$-gpr-open implies $F \subseteq \mathcal{I}$-pint(A)$= A \cap int^*(cl^*(A)$ $= (U \cap V) \cap int^*(cl^*(A))$=$(U \cap V) \cap int^*(cl^*(U \cap V)) \subseteq (U \cap V) \cap int^*(cl^*(U) \cap cl^*(V)) = (U \cap V) \cap int^*(cl^*(U)) \cap int^*(cl^*(V))$. This implies, $F \subseteq int^*(cl^*(V)$= $int^*(V)$, since V is an $\mathcal{I}$-t-set. Since F is $\mathcal{I}$-regular closed, U is $\mathcal{I}$-rg-open and $F \subseteq U$, we have $F \subseteq int^*(U)$. Therefore $F \subseteq int^*(U) \cap int^*(V)$= $int^*(U \cap V)$= $int^*(A)$. Hence A is $\mathcal{I}$-rg-open in $(X, \tau, \mathcal{I})$. □

Corollary 3.8 *A subset A of X is $\mathcal{I}$-rg-open in $(X, \tau, \mathcal{I})$if and only if it is both $\mathcal{I}$-$g\alpha^{**}$-open and an $\mathcal{I}$-C_r-set in $(X, \tau, \mathcal{I})$.*

Proof: The proof is similar to the proof of Theorem 3.7. □

Theorem 3.9 *A subset A of X is $\mathcal{I}$-rg-open in $(X, \tau, \mathcal{I})$if and only if it is both $\mathcal{I}$-$g\alpha^{**}$-open and an $\mathcal{I}$-${C_r}^*$-set in $(X, \tau, \mathcal{I})$.*

Proof: Necessity: Assume that A is $\mathcal{I}$-rg-open in $(X, \tau, \mathcal{I})$. Then by propositions 2.3 and 3.2, we have A is both $\mathcal{I}$-$g\alpha^{**}$-open and an $\mathcal{I}$-${C_r}^*$-set in $(X, \tau, \mathcal{I})$.
Sufficiency: Assume that A is $\mathcal{I}$-$g\alpha^{**}$-open and an $\mathcal{I}$-${C_r}^*$-set in $(X, \tau, \mathcal{I})$. Let $F \subseteq A$ and F is $\mathcal{I}$-regular closed in X. Since A is an $\mathcal{I}$-${C_r}^*$-set, $A = U \cap V$, where U is $\mathcal{I}$-rg-open and V is an $\mathcal{I}$-α^*-set. Now, since F is $\mathcal{I}$-regular closed, $F \subseteq U$ and U is $\mathcal{I}$-rg-open, we have $F \subseteq int^*(U)$. Since A is $\mathcal{I}$-$g\alpha^{**}$-open, $F \subseteq \mathcal{I}$-$\alpha int(A)$ =$A \cap int^*(cl^*(int^*(A)))$ =$(U \cap V) \cap int^*(cl^*(int^*(U \cap V)))$ =$(U \cap V) \cap int^*(cl^*(int^*(U) \cap int^*(V))) \subseteq (U \cap V) \cap int^*(cl^*(int^*(U))) \cap int^*(cl^*(int^*(V)))$= $(U \cap V) \cap int^*(cl^*(int^*(U))) \cap int^*(V)$, since V is an $\mathcal{I}$-α^*-set. This implies $F \subseteq int^*(V)$. Therefore $F \subseteq int^*(U) \cap int^*(V)$= $int^*(U \cap V)$=$int^*(A)$. Hence A is $\mathcal{I}$-rg-open in $(X, \tau, \mathcal{I})$. □

Remark 3.10

1. *The notions of $\mathcal{I}$-gpr-open sets and the notions of $\mathcal{I}$-C_r-sets are independent.*
2. *The notions of $\mathcal{I}$-$g\alpha^{**}$-open sets and the notions of $\mathcal{I}$-C_r-sets are independent.*
3. *The notions of $\mathcal{I}$-$g\alpha^{**}$-open sets and the notions of $\mathcal{I}$-${C_r}^*$-sets are independent.*

Example 3.11 *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then the set $\{a, b\}$ is an $\mathcal{I}$-C_r-set but not $\mathcal{I}$-gpr-open.*

Example 3.12 *Let $X = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{d\}\}$. Then the set $\{a, c, d, e\}$ is $\mathcal{I}$-gpr-open but not an $\mathcal{I}$-C_r-set and the set $\{a, b, e\}$ is an $\mathcal{I}$-C_r-set and $\mathcal{I}$-${C_r}^*$-set set but not $\mathcal{I}$-$g\alpha^{**}$-open.*

Example 3.13 *Let $X = \{a, b, c, d, e, f\}$, $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{d\}\}$. Then the set $\{a, c, d, e, f\}$ is $\mathcal{I}$-$g\alpha^{**}$-open but neither an $\mathcal{I}$-C_r-set nor an $\mathcal{I}$-${C_r}^*$-set.*

4. Decompositions of $\mathcal{I}$-rg-continuity

Definition 4.1 *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\mathcal{I}$-rg-continuous(resp. $\mathcal{I}$-$g\alpha^{**}$-continuous and $\mathcal{I}$-gpr-continuous) if $f^{-1}(V)$ is $\mathcal{I}$-rg-open (resp. $\mathcal{I}$-$g\alpha^{**}$-open and $\mathcal{I}$-gpr-open) in $(X, \tau, \mathcal{I})$ for every open set V in (Y, σ).*

Definition 4.2 *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\mathcal{I}$-C_r-continuous(resp. $\mathcal{I}$-${C_r}^*$-continuous) if $f^{-1}(V)$ is an $\mathcal{I}$-C_r-set(resp. $\mathcal{I}$-${C_r}^*$-set) in $(X, \tau, \mathcal{I})$ for every open set V in (Y, σ).*

Proposition 4.3 *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following hold:*

1. *An $\mathcal{I}$-rg-continuous function is $\mathcal{I}$-$g\alpha^{**}$-continuous.*
2. *An $\mathcal{I}$-$g\alpha^{**}$-continuous function is $\mathcal{I}$-gpr-continuous.*
3. *An $\mathcal{I}$-rg-continuous function is $\mathcal{I}$-gpr-continuous.*

However , Converses are not true as seen from the following examples.

Example 4.4 *Let $X = \{a, b, c, d, e, f\}$, $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, c, d\}, X\}$, $\mathcal{I} = \{\phi, \{d\}$ and $\sigma = \{\phi, \{a, c, d, e, f\}, X\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is $\mathcal{I}$-$g\alpha^{**}$-continuous but not $\mathcal{I}$-rg-continuous.*

Example 4.5 *Let $X = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\mathcal{I} = \{\phi, \{d\}\}$ and $\sigma = \{\phi, \{a, c, d, e\}, X\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is $\mathcal{I}$-gpr-continuous but neither $\mathcal{I}$-rg-continuous nor $\mathcal{I}$-$g\alpha^{**}$-continuous.*

Proposition 4.6 *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following hold:*

1. *An $\mathcal{I}$-C_r-continuous function is $\mathcal{I}$-${C_r}^*$-continuous.*
2. *An $\mathcal{I}$-rg-continuous function is $\mathcal{I}$-C_r-continuous and $\mathcal{I}$-${C_r}^*$-continuous.*

However, Converses are not true as seen from the following examples.

Example 4.7 *Let $X = \{a, b, c, d, e\}$,$\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\mathcal{I} = \{\phi, \{d\}\}$ and $\sigma = \{\phi, \{a, c, d, e\}, Y\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is $\mathcal{I}$-${C_r}^*$-continuous but not $\mathcal{I}$-C_r-continuous.*

Example 4.8 *Let $X = \{a, b, c, d\}$,$\tau = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, X\}$,$\mathcal{I} = \{\phi, \{c\}\}$and $\sigma = \{\phi, \{a, b, c\}, Y\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is $\mathcal{I}$-C_r-continuous and $\mathcal{I}$-${C_r}^*$-continuous but not $\mathcal{I}$-rg-continuous.*

Theorem 4.9 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is $\mathcal{I}$-rg-continuous.*
2. *f is $\mathcal{I}$-gpr-continuous and $\mathcal{I}$-C_r-continuous.*
3. *f is $\mathcal{I}$-$g\alpha^{**}$-continuous and $\mathcal{I}$-C_r-continuous.*
4. *f is $\mathcal{I}$-$g\alpha^{**}$-continuous and $\mathcal{I}$-C_r^*-continuous.*

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