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On Wave Equations Without Global a Priori Estimates*

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ABSTRACT: We investigate the existence and uniqueness of weak solution for a mixed problem for wave operator of the type:

$$L(u) = \frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho - f, \quad \rho > 1.$$

The operator is defined for real functions $u = u(x,t)$ and $f = f(x,t)$ where $(x,t) \in Q$ a bounded cylinder of $\mathbb{R}^{n+1}$.

The nonlinearity $|u|^\rho$ brings serious difficulties to obtain global a priori estimates by using energy method. The reason is because we have not a definite sign for $\int_\Omega |u|^\rho \, u \, dx$. To solve this problem we employ techniques of L. Tartar [16], see also D.H. Sattinger [12] and we succeed to prove the existence and uniqueness of global weak solution for an initial boundary value problem for the operator $L(u)$, with restriction on the initial data u_0, u_1 and on the function f. With this restriction we are able to apply the compactness method and obtain the unique weak solution.

Key Words: Wave operator, Compactness method, Wave equations, Galerkin's method.

Contents

1. Introduction

Motivated by a nonlinear theory of measons field, cf. L.I. Schiff [14], K. Jörgens [6], [7], initiated the investigation, from mathematical point of view, of a nonlinear model for partial differential equations of the type:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + F'\big(|u|^2\big)u = 0, \tag{1.1}$$

for a real function $u = u(x,t)$, $x \in \mathbb{R}^n$ and $t \geq 0$.

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* Dedicated to Jacques Louis Lions (1928-2001), (in memorium)

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With restriction on the function $F\colon \mathbb{R} \to \mathbb{R}$ and on the initial conditions $u(x,0)$, $\dfrac{\partial u}{\partial t}(x,0)$, K. Jörgens [6] proved existence and uniqueness of solution for a initial value problem for (1.1). When

$$F(s) = \mu^2 s + \frac{1}{2}\,\eta^2\, s^2,$$

then (1.1) has the form:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \mu^2 u + \eta^2 |u|^2\, u, \tag{1.2}$$

which is one of the models contained in the nonlinear theory, proposed by L.I. Schiff [14], see also K. Jörgens [7].

Motivated by K. Jörgens [6] and [7], J.L. Lions and W.A. Strauss [8] initiated and developed a large field of research on nonlinear evolutions equations, including K. Jörgens model. See also, I.E. Segal [13], F.E. Browder [3], J.A. Goldstein [4] and [5], L.A. Medeiros [11], W.A. Strauss [15] and Von Wahl [17].

Let us fixe our attention on J.L. Lions [10] ch. 1, where he investigated the initial boundary value problem:

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho\, u = f \quad \text{in} \quad Q, \quad \rho > 0 \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_0(x), \;\; \dfrac{\partial u}{\partial t}(x,0) = u_0(x), \;\; x \in \Omega. \end{array}\right. \tag{1.3}$$

By Ω we denote a bounded connected open set of $\mathbb{R}^n$ with C^2 boundary Γ. The points of $\mathbb{R}^n$ are represented by $x = (x_1, \ldots, x_n)$. The cylinder $\Omega \times (0,T)$ of $\mathbb{R}^{n+1}$ is denoted by Q, that is, $Q = \Omega \times (0,T)$, $T > 0$. Thus, we denote by $u = u(x,t)$, for $(x,t) \in Q$, a real function defined in Q. With Σ we represent the lateral boundary of Q, that is, $\Sigma = \Gamma \times (0,T)$.

Under strong hypothesis on f, u_0 and u_1, it was proved existence of weak solutions for (1.3), cf. J.L. Lions [10] or J.L. Lions and W.A. Strauss [8]. The uniqueness of weak solutions was proved when

$$0 < \rho < \frac{2}{n-2}\,\cdot$$

Remark 1.1 *The type of nonlinearity $|u|^\rho\, u$, $\rho > 0$, in (1.3), is fundamental when we apply the energy method, and the case u^2 is not included. Thus, J.L. Lions [10] investigated the case u^2, in (1.3), by an argument idealized by D.H. Satinger [12]. L. Tartar analysed the case u^2 by another method, look Section 2 of the present paper.*

In this paper we investigate the problem (1.3) but with nonlinearity of the type $|u|^\rho$, $\rho > 1$, after a remark of N.A. Larkin, UEM, Paraná-Brasil, personal communication.

Therefore, we plan, in this paper, to investigate the initial boundary value problem:

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho = f \quad \text{in} \quad Q, \quad \rho > 1, \\ u = 0 \quad \text{on} \quad \Sigma, \\ u(x,0) = u_0(x), \ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega. \end{array}\right. \tag{1.4}$$

This problem with acoustic boundary conditions on a part of Σ and the dissipative term $\beta u'$ in the equation $(1.4)_1$, was investigated by G. Antunes et all [1].

In Section 2 we prove the existence and uniqueness of local weak solution for (1.4), that is, the solution is defined only for $0 \le t \le T_0$, T_0 fixed. In Section 3 with a restriction on the size of u_0, u_1 and f, we prove the existence of global weak solution for (1.4), that is, the solution is defined for all $0 \le t < \infty$. In both cases we must have $1 < \rho < \dfrac{n}{n-2}$ if $n \ge 3$ and $\rho > 1$ if $n = 1, 2$. We also can prove uniqueness of weak solutions as in J.L. Lions [10].

2. Local solutions

We observe that all derivatives we consider are in the sense of the theory of distributions. We employ the notation $L^p(\Omega)$, $1 \le p \le \infty$, $H^m(\Omega)$, $m \in \mathbb{N}$, for Lebesgue and Sobolev spaces, respectively. We also employ the notation $L^p(0,T;X)$, $1 \le p \le \infty$ where X is a Banach space.

Theorem 2.1 *(Local Solution). Suppose $\rho > 1$ if $n = 1$ or 2 and $1 < \rho < \dfrac{n}{n-2}$ if $n \ge 3$. Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$ be given. Then, there exist T_0, with $0 < T_0 < T$, and a unique function $u\colon \Omega \times [0,T_0) \to \mathbb{R}$ in the class:*

$$u \in L^\infty(0,T_0;H_0^1(\Omega)), \quad u' \in L^\infty(0,T_0;L^2(\Omega)),$$

which is a weak solution of the initial boundary value problem (1.4).

Remark 2.2 *By Sobolev's embeding theorem, we have:*

$$H^1(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{for} \quad q = \frac{2n}{n-2}, \ n \ge 3.$$

If $n = 1$ we have continuous functions and when $n = 2$ the embeding of $H^1(\Omega)$ in $L^q(\Omega)$ holds for any real number $q \ge 1$.

We need in the proof of Theorem 2.1 the embeding of $H^1(\Omega)$ in the spaces $L^{2\rho}(\Omega)$ and $L^{\rho+1}(\Omega)$. Thus, we fixe $1 < \rho < \dfrac{n}{n-2}\cdot$ If $n = 3$, we consider $\rho = 2$

and we have the case treated by L. Tartar [16]. Since $2\rho < \dfrac{2n}{n-2} = q$, we have $L^q(\Omega) \hookrightarrow L^{2\rho}(\Omega)$. Also, $L^\rho(\Omega) \hookrightarrow L^{\rho+1}(\Omega)$, because $\rho > 1$. Thus if $1 < \rho < \dfrac{n}{n-2}$ and $n \geq 3$, we obtain

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{2\rho}(\Omega) \hookrightarrow L^{\rho+1}(\Omega).$$

■

Proof of Theorem 2.1. Since $H_0^1(\Omega)$ is separable it has a "Hilbertien" basis, represented by $w_1, w_2, \ldots, w_n, \ldots$ (cf. H. Brezis [2]). Denote by $V_m = [w_1, w_2, \ldots, w_m]$ the subspace of dimension m of $H_0^1(\Omega)$ generated by $w_1, w_2, \ldots, w_m$. If $u_m(t) \in V_m$, it has the representation:

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \,.$$

The approximate system for the Galerkin method consists in the following scheme:

$$\left| \begin{array}{l} (u_m'(t), w) + a(u_m(t)) + (|u_m(t)|^\rho, w) = (f(t), w), \text{ for all } w \in V_m \,, \\ u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega), \\ u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega). \end{array} \right. \tag{2.1}$$

Here, we employ the notation $(\ ,\ )$ for the inner product in $L^2(\Omega)$ and $a(u,v)$ for the Dirichlet form:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u, v \in H_0^1(\Omega).$$

Observe that (2.1), for each fixed $m \in \mathbb{N}$, is a system of nonlinear ordinary differential equations. It has a local solution $u_m = u_m(t)$ for $0 < t < t_m < T$. We will prove the existence of $0 < T_0 < T$ such that (2.1) has a solution $u_m(t)$ for $0 \leq t \leq T_0$. Moreover, we obtain uniform estimates for $u_m(t)$ in $[0, T_0]$, which permits to pass to the limits when $m \to \infty$, obtaining a function $u(\,\cdot\,, t)$ which is weak solution of (1.4). We need estimates for $u_m(t)$.

Estimate 2.3 *Set $w = u_m'(t) \in V_m$ in (2.1). We obtain:*

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}||u_m(t)||^2 + \big(|u_m(t)|^\rho, u_m'(t)\big) = \big(f(t), u_m'(t)\big).$$

Note that $|\cdot|$ and $||\cdot||$ are the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Observe, also, that $|u_m(t)|^\rho$ is the absolute value of $u_m(t)$, power $\rho > 1$.

Thus, we have:

$$\frac{1}{2}\frac{d}{dt}\left[|u_m'(t)|^2 + ||u_m(t)||^2\right] \leq \left|\big(|u_m(t)|^\rho, u_m'(t)\big)\right| + |(f(t), u_m'(t))|. \tag{2.2}$$

Analysis of the right hand side of (2.2)

By Cauchy-Schwarz inequality:

$$\left|\left(|u_m(t)|^\rho, u_m'(t)\right)\right| \le \left||u_m(t)|^\rho\right|_{L^2(\Omega)} \cdot \left|u_m'(t)\right|_{L^2(\Omega)}.$$

Observe that:

$$\left||u_m(t)|^\rho\right|_{L^2(\Omega)} = \left(\int_\Omega |u_m(t)|^{2\rho}\, dx\right)^{1/2} = \left|u_m(t)\right|^\rho_{L^{2\rho}(\Omega)}.$$

From Remark 2.2, we obtain:

$$|u_m(t)|_{L^{2\rho}(\Omega)} \le C_0\, ||u_m(t)||.$$

Thus,

$$\left|\left(|u_m(t)|^\rho, u_m'(t)\right)\right| \le |u_m(t)|^\rho_{L^{2\rho}(\Omega)} \cdot |u_m'(t)| \le C_0^\rho\, ||u_m(t)||^\rho \cdot |u_m'(t)|.$$

Returning to (2.2), observing the above inequality, we have:

$$\frac{1}{2}\frac{d}{dt}\left[|u_m'(t)|^2 + ||u_m(t)||^2\right] \le C_0^\rho\, ||u_m(t)||^\rho \cdot |u_m'(t)| + |f(t)| \cdot |u_m'(t)|. \tag{2.3}$$

Set

$$\varphi_m(t) = \frac{1}{2}\, |u_m'(t)|^2 + \frac{1}{2}\, ||u_m(t)||^2. \tag{2.4}$$

We obtain

$$|u_m'(t)| \le \sqrt{2\varphi_m(t)} \text{ and } \left||u_m(t)|\right| \le \sqrt{2\varphi_m(t)}. \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3) we get:

$$\varphi_m'(t) \le |f(t)|\, (2\varphi_m(t))^{1/2} + C_0^\rho\, (2\varphi_m(t))^{\frac{\rho+1}{2}}. \tag{2.6}$$

We can supppose $\varphi_m(t)^{1/2} \le \varphi_m(t)^{\frac{\rho+1}{2}}$, otherwise we consider $\varphi_m(t)+1$ instead of $\varphi_m(t)$. Therefore, from (2.6) we have:

$$\varphi_m'(t) \le \left(|f(t)|\sqrt{2} + 2^{\frac{\rho+1}{2}}\, C_0^\rho\right) (\varphi_m(t))^{\frac{\rho+1}{2}},$$

or

$$\varphi_m(t)^{-(\frac{\rho+1}{2})}\, \varphi_m'(t) \le \sqrt{2}\, |f(t)| + 2^{\frac{\rho+1}{2}}\, C_0^\rho, \text{ for } 0 \le t < t_m. \tag{2.7}$$

Since

$$\varphi_m(t)^{-(\frac{\rho+1}{2})}\, \varphi_m'(t) = \frac{d}{dt}\left[\frac{2}{1-\rho}\, \varphi_m(t)^{\frac{1-\rho}{2}}\right]$$

and $1-\rho<0$, integrating (2.7) we get:

$$\varphi_m(t)^{-(\frac{\rho-1}{2})} \ge \varphi_m(0)^{-(\frac{\rho-1}{2})} - \left(\frac{\rho-1}{2}\right)\sqrt{2}||f|| - \left(\frac{\rho-1}{2}\right) 2^{\frac{\rho+1}{2}}\, tC_0^\rho,$$

where

$$||f|| = ||f||_{L^1(0,T;L^2(\Omega))} = \int_0^T |f(s)|\,ds.$$

Observe that $\varphi_m(0) = \dfrac{1}{2}\,|u_{1m}|^2 + \frac{1}{2}\,||u_{0m}||^2$ is a bounded sequence of positive real numbers.
Thus,

$$\varphi_m(0) \le A \quad \text{or} \quad \varphi_m(0)^{-1} \ge A^{-1}.$$

Since $\dfrac{\rho-1}{2} > 0$, we have:

$$\varphi_m(0)^{-\frac{\rho-1}{2}} \ge A^{-\frac{\rho-1}{2}}.$$

Therefore, we get:

$$\varphi_m(t)^{-\frac{\rho-1}{2}} \ge A^{-\frac{\rho-1}{2}} - \left(\frac{\rho-1}{2}\right)\sqrt{2}||f|| - \left(\frac{\rho-1}{2}\right) 2^{\frac{\rho+1}{2}}\, t\, C_0^{\rho}\,. \tag{2.8}$$

Choosing $A > 0$, sufficiently large such that

$$A^{-\frac{\rho-1}{2}} - \left(\frac{\rho-1}{2}\right)\sqrt{2}\,||f|| > 0,$$

and defining

$$T^* = \frac{1}{\left(\frac{\rho-1}{2}\right)2^{\frac{\rho+1}{2}}\,C_0^{\rho}}\left(A^{-\frac{\rho-1}{2}} - \left(\frac{\rho-1}{2}\right)\sqrt{2}||f||\right) > 0,$$

we can see that

$$A^{-\frac{\rho-1}{2}} - \left(\frac{\rho-1}{2}\right)\sqrt{2}||f|| - \left(\frac{\rho-1}{2}\right) 2^{\frac{\rho+1}{2}}\, t\, C_0^{\rho} > 0,$$

for all $0 \le t < T^*$.

Let T_0 be fixed such that $0 < T_0 < T^*$. Hence, from (2.8) we obtain

$$\varphi_m(t)^{\frac{\rho-1}{2}} \le \frac{1}{A^{-\frac{\rho-1}{2}} - \left(\frac{\rho-1}{2}\right)\sqrt{2}\,||f|| - \left(\frac{\rho-1}{2}\right) 2^{\frac{\rho+1}{2}}\,C_0^{\rho}\,T_0}\,, \forall\, t \in [0, T_0].$$

This inequality implies that:

$$\begin{aligned} &(u_m)_{m\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0, T_0; H_0^1(\Omega)) \\ &(u'_m)_{m\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0, T_0; L^2(\Omega)). \end{aligned}$$

The above estimates are sufficient to pass the limit in the approximate system (2.1) as $m \to \infty$. Note also that, as in J. Lions [10], we apply a compacity argument. To pass the limit in the nonlinear term $|u_m(t)|^\rho$ we apply Lemma 3.2 in J. Leray and J.L. Lions [9].

Remark 2.4 *About uniqueness of local weak solution, given by Theorem 2.1, we can apply the same argument of J.L. Lions [10]. In fact, we have the restriction* $1 < \rho < \dfrac{n}{n-2}$ *or* $0 < \rho - 1 < \dfrac{2}{n-2}\cdot$ *Thus* $0 < (\rho-1)n < \dfrac{2n}{n-2} = q$ *and this is the condition we need to apply Hölder's inequality.*

We also have $u(x,0) = u_0(x)$ and $u'(x,0) = u_1(x)$ for $x \in \Omega$. The method is the same as J.L. Lions and W.A. Strauss [8].

3. Global solutions

In this section we restrict the size of u_0, u_1 and f in order to obtain global estimates for approximate solutions u_m given in Section 2, system (2.1). These estimates permit us to obtain weak solution for (1.4), defined for all $0 \le t < \infty$ and $x \in \Omega$.

Theorem 3.1 *(Global Solutions). Let* ρ *and* n *be as in Theorem 2.1. For each* $(u_0, u_1, f) \in H_0^1(\Omega) \times L^2(\Omega) \times L^1(0,\infty; L^2(\Omega))$ *we set:*

$$\begin{aligned}\gamma = \Big(&\frac{1}{2}\,|u_1|^2 + \frac{1}{2}\,||u_0||^2 \\ &+ \frac{1}{\rho+1}\int_\Omega |u_0(x)|^\rho\, u_0(x)\,dx + \frac{1}{2}\,||f||\Big)\big(1 + ||f||e^{||f||}\big),\end{aligned} \tag{3.1}$$

where $||f|| = ||f||_{L^1(0,\infty;L^2(\Omega))} = \displaystyle\int_0^\infty |f(s)|\,ds$. *If*

$$0 < ||u_0|| < \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}, \tag{3.2}$$

with C_0 *the constant of embeding of* $H_0^1(\Omega)$ *in* $L^{2\rho}(\Omega)$, $L^{\rho+1}(\Omega)$, *and*

$$\gamma < \frac{1}{2}\left(\frac{1}{C_0}\right)^{\frac{2(\rho+1)}{\rho-1}}, \tag{3.3}$$

then, there exists a unique function $u\colon \Omega \times [0,\infty) \to \mathbb{R}$, *in the class:*

$$u \in L^\infty_{loc}(0,\infty; H_0^1(\Omega)), \quad u' \in L^\infty_{loc}(0,\infty; L^2(\Omega)),$$

which is a weak solution of (1.4).

Proof: We will obtain global estimates for $u_m(t)$, solution of the approximate system (2.1), under the assumptions (3.2) and (3.3) for u_0, u_1 and f. □

Estimate 3.2 *Set $w = u'_m(t)$ in (2.1). We obtain:*

$$\frac{1}{2}\,\frac{d}{dt}\,|u'_m(t)|^2 + \frac{1}{2}\,\frac{d}{dt}\,||u_m(t)||^2 + \int_\Omega |u_m(x,t)|^\rho\, u'_m(x,t)\,dx = (f(t), u'_m(t)).$$

Substituting

$$\int_\Omega |u_m(x,t)|^\rho\, u'_m(x,t)\,dx = \frac{1}{\rho+1}\,\frac{d}{dt}\int_\Omega |u_m(x,t)|^\rho\, u_m(x,t)\,dx,$$

it follows:

$$\frac{d}{dt}\left[\frac{1}{2}\,|u'_m(t)|^2 + \frac{1}{2}\,||u_m(t)||^2 + \frac{1}{\rho+1}\int_\Omega |u_m(x,t)|^\rho\, u_m(x,t)\,dx\right] = \big(f(t), u'_m(t)\big). \tag{3.4}$$

Observe that for $T > 0$ arbitrarily fixed, there exist $T_0 \in (0,T)$ such that (3.4) holds for all $0 \le t \le T_0$ by Theorem 2.1.

Integrating (3.4) on $[0,t]$ for $0 \le t \le T_0$, we obtain:

$$\frac{1}{2}\,|u'_m(t)|^2 + \frac{1}{2}\,||u_m(t)||^2 + \frac{1}{\rho+1}\int_\Omega |u_m(x,t)|^\rho\, u_m(x,t)\,dx \le$$
$$\le \frac{1}{2}\,|u_1|^2 + \frac{1}{2}\,||u_0||^2 + \frac{1}{\rho+1}\int_\Omega |u_0(x)|^\rho\, u_0(x)\,dx + \int_0^t |f(s)|\,|u'_m(s)|\,ds,$$

which can be written as:

$$\frac{1}{2}\,|u'_m(t)|^2 + J\big(u_m(t)\big) \le \frac{1}{2}\,|u_1|^2 + J(u_0) + \int_0^t |f(s)|\,|u'_m(s)|\,ds, \tag{3.5}$$

where $J\colon H_0^1(\Omega) \to \mathbb{R}$ is defined by:

$$J(u) = \frac{1}{2}\,||u||^2 + \frac{1}{\rho+1}\int_\Omega |u|^\rho\, u\,dx. \tag{3.6}$$

The main question, in this point of the proof, is to show that under the assumptions (3.2) and (3.3), we can control the sign of $J(u)$, for $u = u_m(\,\cdot\,,t)$ approximate solution of (2.1), $0 \le t \le T_0$ and at $u = u_0$, in the inequality (3.5).

We note that:

$$\left|\int_\Omega |u|^\rho\, u\,dx\right| \le \int_\Omega |u|^{\rho+1}\,dx = |u|^{\rho+1}_{L^{\rho+1}(\Omega)} \le C_0^{\rho+1}\,||u||^{\rho+1},$$

where in the last inequality we employed $H_0^1(\Omega) \hookrightarrow L^{\rho+1}(\Omega)$, cf. Remark 2.2.

Thus,

$$\int_\Omega |u|^\rho\, u\,dx \ge -C_0^{\rho+1}\,||u||^{\rho+1}.$$

We go back to $J(u)$ and we get:

$$J(u) \ge \frac{1}{2}\,||u||^2 - \frac{C_0^{\rho+1}}{\rho+1}\,||u||^{\rho+1}\,, \tag{3.7}$$

which we employ for $u = u_m(\cdot\,, t)$ and $u = u_0$.

Whence, the sign of both sides of (3.5) depends of the sign of the function:

$$P(\lambda) = \frac{1}{2}\,\lambda^2 - \frac{C_0^{\rho+1}}{\rho+1}\,\lambda^{\rho+1} \text{ for } \lambda \geq 0.$$

Analysis of $P(\lambda)$, $\lambda \geq 0$

- $P(\lambda)$ has zeros at $\lambda_0 = 0$ with order two and at $\lambda_1 = \left(\dfrac{\rho+1}{2C_0^{\rho+1}}\right)^{\frac{1}{\rho-1}}$.
- The derivative of $P(\lambda)$ has zeros at $\lambda_0 = 0$ and $\lambda_1 = \left(\dfrac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}$. Moreover $P(\lambda)$ is increasing in $0 < \lambda < \left(\dfrac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}$.
- $P(\lambda)$ has a maximum at $\lambda_M = \left(\dfrac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}$, and its maximum value in the interval $\left[0, \left(\dfrac{\rho+1}{C_0^{\rho+1}}\right)^{\frac{1}{\rho-1}}\right]$ is

$$\frac{\rho-1}{2(\rho+1)}\left(\frac{1}{C_0}\right)^{\frac{2(\rho+1)}{\rho-1}}.$$

The approximate graphic of $P(\lambda)$ is:

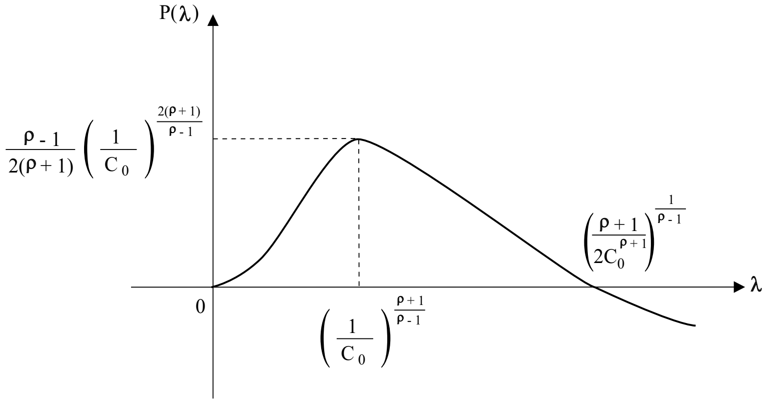

Fig. 1

Now, let us go back to (3.5). By hypothesis (3.2) and inequality (3.7) we have:

$$J(u_0) \geq P(||u_0||) > 0,$$

then the right hand side of (3.5) is positive.

For the sign of $J(u_m(t))$ in the left hand side of (3.5) we need the following result:

Lemma 3.3 *Suppose* u_0 , u_1 *and* γ *satisfying the conditions (3.2) and (3.3) of Theorem 3.1. Then the approximate solution* $(u_m)_{m\in\mathbb{N}}$ *satisfies*

$$||u_m(t)|| < \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}, \text{ for all } t \in [0, T_0] \text{ and } m \in \mathbb{N}. \tag{3.8}$$

Proof: We argue by contradiction method. In fact, suppose there exists $m_0 \in \mathbb{N}$ and some $t \in (0, T_0]$ such that

$$||u_{m_0}(t)|| > \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}. \tag{3.9}$$

□

We know that $u_m(0) = u_{0m} \to u_0$ in $H_0^1(\Omega)$, then $||u_m(0)|| \to ||u_0||$. Thus

$$0 < ||u_{m_0}(0)|| \leq ||u_0||.$$

By (3.2) we have

$$0 < ||u_{m_0}(0)|| \leq ||u_0|| < \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}.$$

Since $||u_{m_0}(t)||$ is continuous in $[0, T_0]$, there exists $t_0 \in (0, T_0)$ such that

$$0 < ||u_{m_0}(t)|| < \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}, \text{ for all } 0 \leq t < t_0\,. \tag{3.10}$$

Now, we consider the subset τ of $(0, T_0)$ defined by:

$$\tau = \left\{ t \in (0, T_0); ||u_{m_0}(t)|| \geq \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}} \right\}.$$

Properties of τ

- It is not empty, because of (3.9).
- It is a closet set because the function $||u_{m_0}(t)||$ is continuous on $[0, T_0]$.
- It is bounded below by (3.10).

Thus, by the properties of τ, it has a minimum $t^* \in (0, T_0)$, which satisfies

$$\left| \begin{array}{l} ||u_{m_0}(t)|| < \left(\dfrac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}} \quad \text{for all } 0 \leq t < t^* \\ ||u_{m_0}(t^*)|| = \left(\dfrac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}} . \end{array} \right. \tag{3.11}$$

Now let us go back to (3.4) and integrate on $(0, t^*)$. We obtain:

$$\frac{1}{2}\left|u'_m(t^*)\right|^2 + J(u_m(t^*)) \leq \frac{1}{2}|u_1|^2 + J(u_0) + \int_0^{t^*} |f(s)|\,|u'_{m_0}(s)|\,ds. \tag{3.12}$$

Observe that

$$J(u_{m_0}(t^*)) \geq P(u_{m_0}(t^*)) > 0,$$

by $(3.11)_2$, because at $\left(\dfrac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}$, $P(\lambda)$ has a maximum strictly positive. Then, it implies that $J(u_{m_0}(t^*)) > 0$, and the left hand side of (3.12) is strictly positive.

Observe that we have:

$$\int_0^{t^*} |f(s)|\,|u'_{m_0}(s)|ds \leq \frac{1}{2}\int_0^{\infty} |f(s)|\,ds + \int_0^{t^*} |f(s)| \left(\frac{1}{2}|u'_{m_0}(s)|^2\right) ds.$$

Going back to (3.12), setting $||f|| = ||f||_{L^1(0,\infty;L^2(\Omega))}$, we obtain:

$$\begin{aligned} \frac{1}{2}|u'_{m_0}(t^*)|^2 + J(u_{m_0}(t^*)) &\leq \frac{1}{2}|u_1|^2 + J(u_0) \\ &+\frac{1}{2}||f|| + \int_0^{t^*} |f(s)| \left(\frac{1}{2}|u'_{m_0}(s)^2\right) ds. \end{aligned} \tag{3.13}$$

We need to evaluate $\frac{1}{2}|u'_{m_0}(t^*)|^2$, knowing that $J(u_{m_0}(t^*)) > 0$.

From (3.13) we have:

$$\frac{1}{2}|u'_{m_0}(t)|^2 \leq \frac{1}{2}|u_1|^2 + J(u_0) + \frac{1}{2}||f|| + \int_0^{t^*} |f(s)| \left(\frac{1}{2}|u'_{m_0}(s)|^2\right) ds. \tag{3.14}$$

Note that (3.14) is an inequality of the type:

$$\varphi(t) \leq K + \int_0^t a(s)\varphi(s)\,ds,$$

with $a(s) = |f(s)| \in L^1(0,\infty)$, $\varphi(t) = \frac{1}{2}|u'_{m_0}(t)|$ and K the positive constant $\frac{1}{2}|u_1|^2 + J(u_0) + \frac{1}{2}||f||$.

From the APPENDIX, we have:

$$\frac{1}{2}\,|u'_{m_0}(t^*)|^2 = \varphi(t^*) \leq Ke^{||f||}.$$

Thus,

$$\int_0^{t^*} |f(s)| \left(\frac{1}{2}\,|u'_{m_0}(s)|^2\right) ds \leq Ke^{||f||}\,||f||.$$

Substituting in (3.13) we get:

$$\begin{aligned}
&\frac{1}{2}\,|u'_{m_0}(t^*)|^2 + J(u_{m_0}(t^*)) \leq \\
&\leq \left(\frac{1}{2}\,|u_1|^2 + J(u_0) + \frac{1}{2}\,||f||\right)\left(1 + ||f||e^{||f||}\right) \\
&= \gamma < \frac{1}{2}\left(\frac{1}{C_0}\right)^{\frac{2(\rho+1)}{\rho-1}}.
\end{aligned} \tag{3.15}$$

From (3.15), since $J(u_{m_0}(t^*)) > 0$, we get

$$\frac{1}{2}\,||u_{m_0}(t^*)||^2 \leq \frac{1}{2}\,|u'_{m_0}(t)|^2 + J(u_{m_0}(t^*)) < \frac{1}{2}\left(\frac{1}{C_0}\right)^{\frac{2(\rho+1)}{\rho-1}},$$

which gives:

$$||u_{m_0}(t^*)|| < \left(\frac{1}{C_0}\right)^{\frac{\rho+1}{\rho-1}}.$$

This is a contradiction with $(3.11)_2$. Thus the Lemma 3.3 is proved. ■

By Lemma 3.3 we have $J(u_m(t)) \geq P(u_m(t)) > 0$ for all $t \in [0, T_0]$ and all $m \in \mathbb{N}$. It follows from (3.5) that:

$$|u'_m(t)| + ||u_m(t)|| < C,$$

C a constant independent of m and T_0.

The functions $u'_m(t)$ and $u_m(t)$ are continuous on $[0, T_0]$. Thus the extension to $[0, T]$ for all real number $T > 0$, permites to obtain

$$\left|\begin{array}{lll} (u_m)_{m\in\mathbb{N}} & \text{is bounded in} & L^\infty(0, T; H_0^1(\Omega)) \\ (u'_m)_{m\in\mathbb{N}} & \text{is bounded in} & L^\infty(0, T; L^2(\Omega)) \end{array}\right. \tag{3.16}$$

for all $0 < T < \infty$ or $L^\infty_{\text{loc}}(0, \infty; \cdot\,)$.

Thus, (3.16) are sufficient to pass the limit in the approximate problem (2.1) to obtain global weak solution for (1.4). For the uniqueness and the initial data we have the same remark done for local solution. ■

4. Appendix

If $\varphi(t) = \dfrac{1}{2}\,|u_m'(t)|^2$ we obtained the differential inequality:

$$\varphi(t) \leq K + \int_0^t |f(s)|\,\varphi(s)\,ds.$$

Then

$$\frac{|f(t)|\,\varphi(t)}{K + \int_0^t |f(s)|\,\varphi(s)\,ds} \leq |f(t)|,$$

with

$$\frac{d}{dt}\left(K + \int_0^t |f(s)|\,\varphi(s)\,ds\right) = |f(t)|\,\varphi(t).$$

Integrating the above differential inequality we have:

$$\int_0^t \frac{|f(\xi)|\,\varphi(\xi)d\xi}{K + \int_0^\xi |f(s)|\,\varphi(s)\,ds} \leq \int_0^t |f(s)|\,ds,$$

that is

$$lg\left(K + \int_0^\xi |f(s)|\,\varphi(s)\,ds\right)\Bigg|_{\xi=0}^{\xi=t} \leq ||f||$$

or

$$lg\left(K + \int_0^t |f(s)|\,\varphi(s)\,ds\right) - lg\,K \leq ||f||.$$

It implies that

$$K + \int_0^t |f(s)|\,\varphi(s)\,ds \leq K\,e^{||f||},$$

that is

$$\frac{1}{2}\,|u_m'(t)|^2 = \varphi(t) \leq K\,e^{||f||}.$$

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