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# Characterization of Spacelike Biharmonic Curves with Timelike Binormal According to Flat Metric in Lorentzian Heisenberg Group $Heis^3$

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ABSTRACT: In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group $Heis^3$. We characterize spacelike biharmonic curves with timelike binormal in terms of their curvature and torsion. Additionally, we determine the parametric representation of the spacelike biharmonic curves with timelike binormal according to flat metric from this characterization.

Key Words: Biharmonic curve, Heisenberg group, Flat metric.

## Contents



## 1. Introduction

The theory of biharmonic maps is an old and rich subject, initially studied due to its implications in the theory of elasticity and uid mechanics. G.B. Airy and J.C. Maxwell were the first to study and express plane elastic problems in terms of the biharmonic equation. Later on, the theory evolved with the study of polyharmonic functions developed by E. Almansi, T. Levi-Civita, M. Nicolaescu.

Let $f:(M,g)\to(N,h)$ be a smooth function between two Riemannian manifolds. The bienergy $E_2(f)$ of $f$ over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))\,dv_g, \tag{1.1}$$

where $\tau(f)=\text{trace}_g\nabla df$ is the tension field of $f$ and $dv_g$ is the volume form of $M$. Using the first variational formula one sees that $f$ is a biharmonic function if and only if its bitension field vanishes identically, i.e.

$$\widetilde{\tau}(f):=-\triangle^f(\tau(f))-\text{trace}_g R^N(df,\tau(f))df=0, \tag{1.2}$$

2000 *Mathematics Subject Classification:* 31B30, 58E20 where

$$\triangle^f = -\text{trace}_g(\nabla^f)^2 = -\text{trace}_g\left(\nabla^f\nabla^f - \nabla^f_{\nabla^M}\right) \tag{1.3}$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN\ )$and $R^N$ is the curvature operator of $(N,h)$ defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group $\text{Heis}^3$. We characterize spacelike biharmonic curves with timelike binormal in terms of their curvature and torsion.

## 2. The Lorentzian Heisenberg Group $\text{Heis}^3$

The Heisenberg group $\text{Heis}^3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z)*(\overline{x},\overline{y},\overline{z}) = (x+\overline{x}, y+\overline{y}, z+\overline{z}-\overline{x}y+x\overline{y}).$$

The identity of the group is $(0,0,0)$ and the inverse of $(x,y,z)$ is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on $\text{Heis}^3$ is

$$g = dx^2 + (xdy+dz)^2 - ((1-x)\,dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2,\mathbf{e}_3] = 0,\quad [\mathbf{e}_3,\mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,\quad [\mathbf{e}_2,\mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = 1,\quad g(\mathbf{e}_3,\mathbf{e}_3) = -1. \tag{2.2}$$

**Proposition 2.1** *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g$, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

*where the $(i,j)$-element in the table above equals $\nabla_{e_i}e_j$ for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1,\mathbf{e}_3)=R(\mathbf{e}_1,\mathbf{e}_2)=R(\mathbf{e}_2,\mathbf{e}_3)=0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

## 3. Spacelike Biharmonic Curves with Timelike Binormal According to Flat Metric in the Lorentzian Heisenberg Group Heis$^3$

An arbitrary curve $\gamma : I \longrightarrow Heis^3$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike curve with timelike binormal and $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa_1\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa_1\mathbf{t}+\kappa_2\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= \kappa_2\mathbf{n}, \end{aligned} \tag{3.1}$$

where $\kappa_1$, $\kappa_2$ are curvature function and torsion function, respectively and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\mathbf{n},\mathbf{n})=1,\ g(\mathbf{b},\mathbf{b})=-1, \\ g(\mathbf{t},\mathbf{n}) &= g(\mathbf{t},\mathbf{b})=g(\mathbf{n},\mathbf{b})=0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t_1\mathbf{e}_1+t_2\mathbf{e}_2+t_3\mathbf{e}_3, \\ \mathbf{n} &= n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3, \\ \mathbf{b} &= b_1\mathbf{e}_1+b_2\mathbf{e}_2+b_3\mathbf{e}_3. \end{aligned}$$

**Theorem 3.1** *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2-\kappa_2^2 &= 0, \\ \kappa_2 &= \text{constant}. \end{aligned} \tag{3.2}$$

**Proof:** Using Equation (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_{\mathbf{t}}^3\mathbf{t}-\kappa_1R(\mathbf{t},\mathbf{n})\mathbf{t} \\ &= (-3\kappa_1'\kappa_1)\mathbf{t}+(\kappa_1''-\kappa_1^3+\kappa_1\kappa_2^2)\mathbf{n}+(2\kappa_2\kappa_1'+\kappa_1\kappa_2')\mathbf{b}-\kappa_1R(\mathbf{t},\mathbf{n})\mathbf{t}. \end{aligned}$$

On the other hand, from Equation (2.4) we get

$$(-3\kappa_1'\kappa_1)\mathbf{t}+(\kappa_1''-\kappa_1^3+\kappa_1\kappa_2^2)\mathbf{n}+(2\kappa_2\kappa_1'+\kappa_1\kappa_2')\mathbf{b}=0. \tag{3.3}$$

From (3.3), we obtain

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= 0, \\ \kappa_2' &= 0. \end{aligned} \tag{3.4}$$

This completes the proof. □

**Corollary 3.2** *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal, then $\gamma$ is a helix.*

**Theorem 3.3** *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of $\gamma$ are*

$$\begin{aligned} x(s) &= \cosh\varphi s + C_1, \\ y(s) &= \frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C] + \sinh[\frac{\kappa_1 s}{\sinh\varphi} + C]] + C_2, \\ z(s) &= -\frac{(-1 + C_1 + \cosh\varphi s)\sinh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C] \\ &\quad + \frac{\sinh^2\varphi\cosh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C] + \cosh[\frac{\kappa_1 s}{\sinh\varphi} + C]] \\ &\quad - \frac{\sinh\varphi(\cosh\varphi s + C_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C]] + C_3, \end{aligned} \tag{3.5}$$

*where $C_1, C_2, C_3$ are constants of integration.*

**Proof:** Assume that $\gamma$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric in the Lorentzian Heisenberg group $\mathrm{Heis}^3$. Since $\gamma$ is spacelike biharmonic , $\gamma$ is a helix. So, without loss of generality, we take the axis of $\gamma$ is parallel to the spacelike vector $\mathbf{e}_1$. Then,

$$g(\mathbf{t}, \mathbf{e}_1) = t_1 = \cosh\varphi, \tag{3.6}$$

where $\varphi$ is constant angle.

Direct computations show that

$$\mathbf{t} = \cosh\varphi\mathbf{e}_1 + \sinh\varphi\sinh\wp\mathbf{e}_2 + \sinh\varphi\cosh\wp\mathbf{e}_3. \tag{3.7}$$

Using above equation and Frenet equations, we obtain

$$\wp = \frac{\kappa_1 s}{\sinh\varphi} + C, \tag{3.8}$$

where $C$ is a constant of integration.

From these we get the following formula

$$\begin{aligned} \mathbf{t} &= \cosh\varphi\mathbf{e}_1 + \sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C]\mathbf{e}_2 \\ &\quad + \sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C]\mathbf{e}_3. \end{aligned} \tag{3.9}$$

Substituting (2.1) in above equation, we get

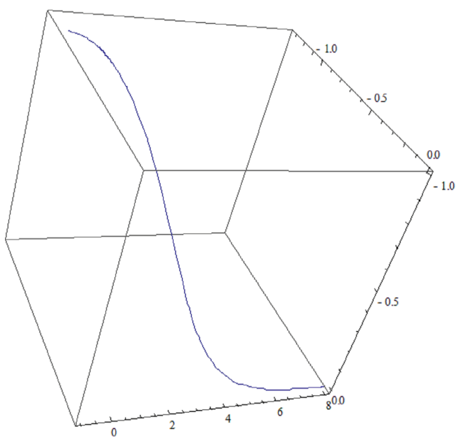
$$\begin{aligned} \mathbf{t} = (&\cosh\varphi, \sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C] + \sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C], \\ &(1-x)\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C] - x\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C]). \end{aligned} \tag{3.10}$$

Now using Equation (3.10) we obtain

$$\begin{aligned} \frac{dx}{ds} &= \cosh\varphi, \\ \frac{dy}{ds} &= \sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C] + \sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C], \\ \frac{dz}{ds} &= (1 - (\cosh\varphi s + \ell_1))\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C] \\ &\quad - (\cosh\varphi s + \ell_1)\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C]. \end{aligned}$$

If we take integrate above system we have Equation (3.5). The proof is completed. □

Using Mathematica in above Theorem, we have following figure.



Using second equation of system (3.2), we express the following Corollary without proof:

**Corollary 3.4** *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then*

$$\kappa_1 = \mp\kappa_2.$$

**Theorem 3.5** *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of $\gamma$ in terms of $\kappa_2$ are*

$$\begin{aligned} x(s) &= \cosh\varphi s + C_1, \\ y(s) &= \mp\frac{1}{\kappa_2}\sinh^2\varphi[\cosh[\mp\frac{\kappa_2 s}{\sinh\varphi}+C]+\sinh[\mp\frac{\kappa_2 s}{\sinh\varphi}+C]]+C_2, \\ z(s) &= \mp\frac{(-1+C_1+\cosh\varphi s)\sinh\varphi}{\kappa_2}\cosh[\mp\frac{\kappa_2 s}{\sinh\varphi}+C] \\ &\quad +\frac{\sinh^2\varphi\cosh\varphi}{\kappa_2^2}[\sinh[\mp\frac{\kappa_2 s}{\sinh\varphi}+C]+\cosh[\mp\frac{\kappa_2 s}{\sinh\varphi}+C]] \\ &\quad \mp\frac{\sinh\varphi(\cosh\varphi s+C_1)}{\kappa_2}\sinh[\mp\frac{\kappa_2 s}{\sinh\varphi}+C]]+C_3, \end{aligned} \tag{3.11}$$

*where $C_1, C_2, C_3$ are constants of integration.*

**Proof:** Using Lemma 3.4 in Equation (3.5), we obtain Equation (3.11). Thus, the proof is completed. □

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