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An Introduction to the Generalized Fractional Integration

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ABSTRACT: The purpose of the present paper is to investigate the generalized fractional integration of the generalized M-series.Some results derived by Saxena and Saigo [13], Samko, Kilbas and Marichev [15] are the special cases of the main results derived in this paper.

Key Words: Wright generalized hypergeometric function , Riemann- Liouville fractional integral operators, generalized Riemann-Liouville and Erdlyi-Kober fractional integral operators, generalized M-series.

Contents

1. Introduction

The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by Mittag-Leffler [5,6] in terms of the power series

$$E_{\alpha}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)}, (\alpha > 0, z \in C) \tag{1.1}$$

A generalization of this series in the following form

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)}, (\alpha, \beta > 0, z \in C) \tag{1.2}$$

has been studied by several authors notably by Mittag-Leffler [5,6], Wiman [3], Agrawal [14], Humbert and Agrawal[12] and Dzrbashjan [7,8,9]. A detailed account of the basic properties of these two functions are given in the third volume of

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Bateman manuscript project [1] and an account of their various properties can be found in [8,14].
The Wright generalized hypergeometric function[4] is given by

$$\Psi_q(z) =_p \Psi_q \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_p, A_p) \\ (\beta_1, B_1, ..., (\beta_q, B_q)) \end{array} ; z \right] = \sum_{r=0}^{\infty} \frac{\Pi_{i=1}^{p} \Gamma(\alpha_i + rA_i) z^r}{\Pi_{j=1}^{q} \Gamma(\beta_j + rB_j) r!} \tag{1.3}$$

$$[A_i > 0 (i = 1, 2, ..., p), B_j > 0 (j = 1, 2, ..., q); \alpha_i \beta_j \in C; 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq 0]$$

It is provided that the Riemann-Liouville fractional integral and derivative of the Wright function is also the Wright function but of greater order. Conditions for the existence of the series (1.3) together with its presentation in terms of the Mellin-Barnes integral and of the H-function were established in [2].
When $A_1 = ... = A_p = B_1 = ... = B_q = 1$, (1)reduces to ${}_pF^q(.)$

$${}_p\Psi_q \left[\begin{array}{c} (\alpha_1, 1), ..., (\alpha_p, 1) \\ (\beta_1, 1, ..., (\beta_q, 1)) \end{array} ; z \right] = \frac{\Pi_{j=1}^{q} \Gamma(\beta_j}{\Pi_{j=1}^{p} \Gamma(\alpha_j)} {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) \tag{1.4}$$

where $p \leq q, |z| < \infty; p = q+1; |z| < 1; p = q+1; |z| = 1, Re(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j) > 0$.
The generalized M-Series [10] is defined as

$${}_pM_q^{\alpha,\beta}(a_1, ..., a_p; b_1, ..., b_q; x) =_p M_q^{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{x^n}{\Gamma(\alpha n + \beta)} \tag{1.5}$$

where $\alpha, \beta \in C, R(\alpha) > 0$ and $(a_i)_n (i = 1, 2, ...p)$are the Pochhammer symbols. further details of this series are given by [10].
Following Section 2 of the book by Samko, Kilbas and Marichev [15], the fractional Riemann-Liouville(R-L) integral operators are given by

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t) dt \tag{1.6}$$

$$I_{-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t - x)^{\alpha - 1} f(t) dt \tag{1.7}$$

where $x > 0, \alpha \in C$ and $R(\alpha) > 0$.

An interesting and useful generalization of the Riemann-Liouville and Erdlyi-Kober fractional integral operators has been introduced by Saigo [11] in terms of Gauss hypergeometric function as given below:

$$(I_{0+}^{\alpha,\beta,\gamma}) f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} {}_2F_1(\alpha + \beta, -\gamma; \alpha; 1 - \frac{t}{x}) f(t) dt \tag{1.8}$$

$$(I_{-}^{\alpha,\beta,\gamma})f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{\alpha-\beta} {}_2F_1(\alpha+\beta, -\gamma; \alpha; 1-\frac{x}{t}) f(t)dt \quad (1.9)$$

where $x \in R_+; \alpha, \beta, \gamma \in C$ and $R(\alpha) > 0$.

2. Left-sided Generalized Fractional Integration of the Generalized M-Series

In this section we derive the left-sided generalized fractional integration formula of the generalized M-Series. The result is presented in the form of a theorem stated below.

Theorem 2.1 *Let $\alpha, \beta, \gamma, \eta \in C$ such that $Re(\alpha) > 0, Re(\eta+\gamma-\beta) > 0, \xi > 0$ and $c \in R$ and $I_{0+}^{\alpha.\beta,\gamma}$ be the left sided operator of the generalized fractional integration then there holds the relation*

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pM_q^{\xi,\eta}[ct^{\xi}]))(x) =$$

$$\frac{x^{\eta-\beta-1}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1...\Gamma(a_p)} \times_{p+2} \Psi_{q+2} \left[\begin{array}{c} (a_1,1),...,(a_p,1),(\eta-\beta+1,\xi),(1,1) \\ (b_1,1),...,(b_q,1)(\eta-\beta,\xi),(\eta+\alpha+1,\xi) \end{array} ; cx^{\xi} \right] \quad (2.1)$$

provided each member of the equation exists.

Proof: By virtue of (1.5) and (1.8), we have

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pM_q^{\xi,\eta}[ct^{\xi}]))(x) =$$

$$\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{t}{x}\right) (t^{\eta-1}{}_pM_q^{\xi,\eta}[ct^{\xi}])dt \quad (2.2)$$

Interchanging the order of integration and summations, evaluating the inner integral with the help of Beta function and using Gauss summation theorem, it becomes

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pM_q^{\xi,\eta}[ct^{\xi}]))(x) =$$

$$\frac{x^{\eta-\beta-1}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)} \times \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)...\Gamma(a_p+k)\Gamma(1+k)\Gamma(\eta+\gamma-\beta+\xi k)(cx^{\xi})^k}{\Gamma(b_1+k)...\Gamma(b_q+k)\Gamma(\eta-\beta+\xi k)\Gamma(\eta+\alpha+\gamma+\xi k)k!} \quad (2.3)$$

Or equivalently

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pM_q^{\xi,\eta}[ct^{\xi}]))(x) =$$

$$\frac{x^{\eta-\beta-1}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1...\Gamma(a_p)} \times_{p+2}\Psi_{q+2}\left[\begin{array}{c}(a_1,1),...,(a_p,1),(\eta-\beta+1,\xi),(1,1)\\(b_1,1),...,(b_q,1)(\eta-\beta,\xi),(\eta+\alpha+1,\xi)\end{array};cx^{\xi}\right] \tag{2.4}$$

This proves theorem(2.1). □

Remark 2.2 *If we put p=q=0 and $\beta\to-\alpha$ in (2.1), we arrive at the well known result [15].*

3. Right-sided Generalized Fractional Integration of the Generalized M-Series

In this section we derive the right-sided generalized fractional integration formula of the generalized M-series. The result is presented in the form of a theorem stated below.

Theorem 3.1 *Let $\alpha,\beta,\gamma,\eta\in C$ such that $Re(\alpha)>0, Re(\alpha+\eta)>max\{-Re(\beta), -Re(\gamma)\}, Re(\beta)\neq Re(\gamma), \xi>0$ and $c\in R$ and $I_{-}^{\alpha,\beta,\gamma}$ be the right-sided operator of the generalized fractional integration then there holds the relation:*

$$(I_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\eta}{}_pM_q^{\xi,\eta}[ct^{-\xi}]))(x)=\frac{x^{-\eta-\alpha-\beta}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)}$$

$$\times_{p+3}\psi_{q+3}\left[\begin{array}{c}(a_1,1),...,(a_p,1),(\alpha+\beta+\eta,\xi),(\alpha+\gamma+\eta,\xi)(1,1)\\(b_1,1),...,(b_q,1),(\eta,\xi),(\alpha+\eta,\xi),(2\alpha+\beta+\gamma+\eta,\xi)\end{array};cx^{-\xi}\right] \tag{3.1}$$

provided each member of the equation exists.

Proof: By using (1.5) and (1.9), we have

$$(I_{-}^{\alpha,\beta,\gamma}(t_p^{\alpha-\eta}M_q^{\xi,\eta}[ct^{-\xi}]))(x)=$$

$$\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t})(t^{-\alpha-\eta}{}_pM_q^{\xi,\eta}[ct^{-\xi}])dt \tag{3.2}$$

Interchanging the order of integration and summations, evaluating the inner integral by the use of Beta function and using Gauss summation theorem, it becomes

$$(I_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\eta}{}_pM_q^{\xi,\eta}[ct^{-\xi}]))(x)=\frac{x^{-\eta-\alpha-\beta}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k)...\Gamma(a_p+k)\Gamma(\alpha+\beta+\eta+\xi k)\Gamma(\alpha+\gamma+\eta+\xi k)(cx^{-\xi})^k}{\Gamma(b_1+k)...\Gamma(b_q+k)\Gamma(\eta-\beta+\xi k)\Gamma(\eta+\alpha+\gamma+\xi k)k!} \tag{3.3}$$

or equivalently

$$(I_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\eta}{}_pM_q^{\xi,\eta}[ct^{-\xi}]))(x)=\frac{x^{-\eta-\alpha-\beta}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)}$$

$$\times_{p+3}\psi_{q+3}\left[\begin{array}{c}(a_1,1),...,(a_p,1),(\alpha+\beta+\eta,\xi),(\alpha+\gamma+\eta,\xi)(1,1)\\(b_1,1),...,(b_q,1),(\eta,\xi),(\alpha+\eta,\xi),(2\alpha+\beta+\gamma+\eta,\xi)\end{array};cx^{-\xi}\right] \quad (3.4)$$

This completes the proof of the theorem(3.1). □

Remark 3.2 *If we put p=q=0 and $\beta \to -\alpha$ in (3.1), we arrive at the well known result [13].*

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