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On Application of Fractional Differintegral Operator to the K_4-Function

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ABSTRACT: The object of this paper is to introduce a new function called K_4-function defined by the author in terms of some special functions, which is an extension of the G-function defined by Lorenzo and Hartley [4] and demonstrate how K_4-function is closely related to another special functions, namely Lorenzo-Hartleyâ's R-function, Robotnov and Hartleyâ's F-function, Mittag-Leffler function, generalized Mittag-Leffler function, Exponential function. The differintegration of that function is also investigated. As special cases most of the results obtained in this paper are believed to be new and include some of the results given earlier by other authors.

Key Words: Differintegration, Fractional calculus, R- and G-functions of Lorenzo-Hartley(L-H)

Contents

1. Introduction and Definitions

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades Fractional Calculus has been applied to almost every field of Mathematics like Special Functions etc., Science, Engineering and Technology. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic Dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory, Image Processing, Non-linear Biological Systems and Astrophysics.

The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by Mittag-Leffler [7] and [8] in terms of the power series

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$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)}, (\alpha > 0) \tag{1.1}$$

A generalization of this series in the following form

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, (\alpha, \beta > 0) \tag{1.2}$$

has been studied by several authors notably by Mittag-Leffler [7] and [8],Wiman [3], Agrawal [17], Humbert and Agrawal [14] and Dzrbashjan [11] [12] [13]. It is shown in [15] that the function defined by (1.1) and (1.2) are both entire functions of order and type A detailed account of the basic properties of these two functions are given in the third volume of Bateman manuscript project [2] and an account of their various properties can be found in [12] and [18].

The multiindex Mittag-Leffler function is defined by Kiryakova [20] by means of the power series

$$E_{(\frac{1}{\rho i}),(\mu i)}(x) = \sum_{n=0}^{\infty} \varphi_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{\prod_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})} \tag{1.3}$$

where $m > 1$ is an integer, and ρ_j and μ_j are arbitrary real numbers.

The multiindex Mittag-Leffler function is an entire function and also gives its asypototic, estimate, order and type see Kiryakova [20].

An interesting generalization of (1.2) is recently introduced by Kilbas and Saigo [1] in terms of a special entire function of the form

$$E_{\alpha,m,l}(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1.4}$$

where

$$c_n = \prod_{i=0}^{n-1} \frac{\Gamma[\alpha(im + l) + 1]}{\Gamma[\alpha(im + l + 1) + 1]}, (n = 0, 1, 2, ...).$$

and an empty product is to be interpreted as unity. Certain properties of this function associated with fractional integrals and derivatives [18].

In 1993, Miller and Ross [10] introduced a function as the basis of the solution of fractional order initial value problem. It is defined as the vth integral of the exponential function, that is,

$$E_x[v, a] = \frac{d^{-v}}{dx^{-v}} e^{ax} = x^v e^{ax} \gamma^*(v, ax) = \sum_{n=0}^{\infty} \frac{a^n x^{n+v}}{\Gamma(\ n + v + 1)}, v \varepsilon C \tag{1.5}$$

where $\gamma^*(v, ax)$ is the incomplete gamma function [10].

The F-function of Robotnov and Hartley [19] is defined by the power series

$$F_q[a,x]=\sum_{n=0}^{\infty}\frac{a^n x^{(n+1)q-1}}{\Gamma((n+1)q)},q>0 \tag{1.6}$$

This function effect the direct solution of the fundamental linear fractional order differential equation.

Recently, the interest in the R- and G-functions of Lorenzo-Hartley [4] [5] and their popularity have sharply increased in view of their important role and applications in Fractional Calculus and related integral and differential equations of fractional order.

The R- and the G-functions(but not the Meijer's G-function) introduced by Lorenzo-Hartley [4] are defined by the power series

$$R_{q,v}[a,c,x]=\sum_{n=0}^{\infty}\frac{a^n(x-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} \tag{1.7}$$

where x$>c\geq 0, q\geq 0, R(q-v)>0$, and

$$G_{\alpha,\beta,\gamma}[a,c,x]=\sum_{n=0}^{\infty}\frac{(\gamma)_n a^n(x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \tag{1.8}$$

where $R(\alpha\gamma-\beta)>0$, and $(\gamma)_n$ is the Pochhammer symbol

$(\gamma)_n=\{{}^{1,n=0}_{\gamma(\gamma+1)....(\gamma+n-1),n\varepsilon N.}$

The present paper is organized as follows; In section 2, we give the definition of the K_4 -function and demonstrate how K_4-function is closely related to another special functions, namely Lorenzo-Hartley's function, Robotnov and Hartley's function, Mittag-Leffler function, generalized Mittag-Leffler function, Exponential function. In section 3, we derive the relation between K_4-function and the operator of differintegral given by Oldham and Spanier and discuss the particular cases.

2. A New Special Function

The K_4-function introduced by the author is defined as follows:

$$K_4^{(\alpha,\beta,\gamma),(a,c):(r;s)}(a_1,...,a_r;b_1,...b_s;x)=K_4^{(\alpha,\beta,\gamma),(a,c):(r;s)}(x) \tag{2.1}$$
$$=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_r)_n}{(b_1)_n...(b_s)_n}\frac{(\gamma)_n a^n(x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)}$$

where $R(\alpha\gamma-\beta)>0$ and $(a_i)_n$(i=1,2,...,r) and $(b_j)_n$(j=1,2,...,s) are the Pochhammer symbols.

The series(2.1) is defined when none of the parameters b_js is a negative integer or zero. If any numerator parameter a_i is a negative integer or zero, then the series terminates to a polynomial in x. From the ratio test it is evident that the series is convergent for all x if r>s + 1. When r = s + 1 and $\mid x\mid=1$, the series can converge in some cases. Let $\Upsilon = \sum_{j=1}^{r} \mathrm{a}_j$ - $\sum_{j=1}^{s} \mathrm{b}_j$. It can be shown that when r = s + 1 the series is absolutely convergent for $\mid x \mid= 1$ if $R(\Upsilon) < 0$, conditionally convergent for $x = -1$ if $0 \leq R(\Upsilon) < 1$ and divergent for $\mid x \mid= 1$ if $1 \leq R(\Upsilon)$.

Relations with another Special functions:

(i) When there is no upper and lower parameters, we get

$$K_4^{(\alpha,\beta,\gamma),(a,c):(0;0)}(-;-;x) = K_4^{(\alpha,\beta,\gamma),(a,c):(0;0)}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n a^n (x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \tag{2.2}$$

which reduces to the G-function defined by Lorenzo and Hartley [4] and denoted by $\mathrm{G}_{\alpha,\beta,\gamma}[a,c,x]$.

(ii) If we put $\alpha = q, \beta = v, \gamma = 1$ in (2.2), we get

$$K_4^{(q,v,1),(a,c):(0;0)}(-;-;x) = K_4^{(q,v,1),(a,c):(0;0)}(x) = \sum_{n=0}^{\infty} \frac{a^n (x-c)^{(n+1)q-v-1}}{\Gamma((n+1)q-v)} \tag{2.3}$$

which reduces to the R-function defined by Lorenzo and Hartley [4] and denoted by $\mathrm{R}_{q,v}[a,c,x]$.

(iii) If we take c=v=0 in (2.3), we get

$$K_4^{(q,0,1),(a,0):(0;0)}(-;-;x) = K_4^{(q,0,1),(a,0):(0;0)}(x) = \sum_{n=0}^{\infty} \frac{a^n x^{(n+1)q-1}}{\Gamma((n+1)q)} \tag{2.4}$$

which reduces to the F-function defined by Robotnov and Hartley [19] and denoted by $\mathrm{F}_q[a,x]$.

(iv) If we set a=q=1 in (2.4), we obtain

$$K_4^{(1,0,1),(1,0):(0;0)}(-;-;x) = K_4^{(1,0,1),(1,0):(0;0)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} \tag{2.5}$$

which is the Mittag-Leffler function [2] $E_1(x)$ or generalized Mittag-Leffler function [2] $E_{1,1}(x)$ or Exponential function [6] e^x.

3. Differintegration of the K_4-Function

In this section we derive the relation between K_4 -function and the operator of differintegral given by Oldham and Spanier. The relation is presented in the form of the theorem as follows:

Theorem 3.1 *Let* $-\infty < \delta < \infty$, *R*$(\alpha\gamma - \beta) > 0, x > c \geq 0$ *and* ${}_c d_x^{\delta}$ *be the operator of differintegral given by Oldham and Spanier then there holds the relation:*

$$ {}_c d_x^{\delta} K_4^{(\alpha,\beta,\gamma),(a,c):(r;s)}(a_1,...,a_r;b_1,...b_s;x) = K_4^{(\alpha,\beta+\delta,\gamma),(a,c):(r;s)}(a_1,...,a_r;b_1,...b_s;x) \tag{3.1}$$

Proof: The δth order$(-\infty < \delta < \infty)$, differintegral oprator defined in the chapter 4 of the book by Oldham and Spanier [9] of the function f(x) is given by

$$_cd_x^\delta f(x) = \frac{df(x)}{d(x-c)^\delta} \tag{3.2}$$

Using (2.1) and (3.2) and interchanging the order of integration and evaluating the inner integral, we arrive at the result

$$_cd_x^\delta K_4^{(\alpha,\beta,\gamma),(a,c):(r;s)}(a_1,...,a_r;b_1,...b_s;x) = K_4^{(\alpha,\beta+\delta,\gamma),(a,c):(r;s)}(a_1,...,a_r;b_1,...b_s;x) \tag{3.3}$$

This shows that differintegral of the K_4 -function is again the K_4 -function with indice $\beta+\delta$.

This completes the proof of the theorem 3.1. □

Particular cases: (i)If we take r=s=0 in (3.1) it reduces to

$$_cd_x^\delta K_4^{(\alpha,\beta,\gamma),(a,c):(0;0)}(-;-;x) = K_4^{(\alpha,\beta+\delta,\gamma),(a,c):(0;0)}(-;-;x) \tag{3.4}$$

Or equivalently

$$_cd_x^\delta G_{\alpha,\beta,\gamma}[a,c,x] = G_{\alpha,\beta+\delta,\gamma}[a,c,x] \tag{3.5}$$

which is the δ th order differintegral of the G-function defined by Lorenzo and Hartley [4].

(ii)Setting $\alpha = q, \beta = v, \gamma = 1$ in (3.5), we obtain

$$_cd_x^\delta G_{q,v,1}[a,c,x] = G_{q,v+\delta,1}[a,c,x] \tag{3.6}$$

Or equivalently

$$_cd_x^\delta R_{q,v}[a,c,x] = R_{q,v+\delta}[a,c,x] \tag{3.7}$$

which is the δ th order differintegral of the R-function defined by Lorenzo and Hartley [4].

(iii)If we let c=v=0 in (3.7), we obtain

$$_0d_x^\delta R_{q,0}[a,0,x] = R_{q,\delta}[a,0,x] \tag{3.8}$$

Or equivalently

$$_0d_x^\delta F_q[a,0,x] = R_{q,\delta}[a,0,x] \tag{3.9}$$

which is the δth order differintegral of the F-function defined by Robotnov and Hartley [19].

(iv)If we set a=q=1 in (3.9), we obtain

$$_0d_x^\delta F_1[1,0,x] = R_{1,\delta}[1,0,x] \tag{3.10}$$

which is the $\delta thorderdifferintegraloftheMittag-Lefflerfunction$ [2]$\mathrm{E}_1(x)$ or generalized Mittag-Leffler function [2] $E_{1,1}(x)$ or Exponential function [6] e^x .

4. Conclusion

It is expected that some of the results derived in this survey may find applications in the solution of certain fractional order differential and integral equations arising problems of physical sciences and engineering areas.

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