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A survey of non-abelian tensor products of groups and related constructions

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ABSTRACT: We report on a group construction in connection with non-abelian tensor products of groups and recent development in non-abelian tensor products and q-tensor products.

Key Words: non-abelian tensor product, polycyclic group, q-tensor product

Contents

1. Introduction

This manuscript is an expanded version of a talk delivered on occasion of the Summer School of Mathematics held at Universidade Estadual de Maringá - UEM, Brazil, on March 1-5, 2010. The authors are grateful to the organizers for their kind invitation to the event. The second named author is also grateful for the worm hospitality offered by his colleagues in the UEM during his stay in Maringá.

The non-abelian tensor product $G \otimes H$ of groups G and H was introduced by Brown and Loday [3,4] following works of Miller [24], Dennis [8] and Lue [21]. It is defined for any pair of groups G and H where each one acts on the other (on right)

$$G \times H \to G,\ (g, h) \mapsto g^h; \quad H \times G \to H,\ (h, g) \mapsto h^g$$

and on itself by conjugation, in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{(h^{g_1})} = \left(\left(g^{g_1^{-1}}\right)^h\right)^{g_1} \quad \text{and} \quad h^{(g^{h_1})} = \left(\left(h^{h_1^{-1}}\right)^g\right)^{h_1}. \tag{1}$$

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In this situation we say that G and H act *compatibly* on each other. The *non-abelian tensor product* $G \otimes H$ is the group generated by all symbols $g \otimes h$, $g \in G$, $h \in H$, subject to the relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

for all $g, g_1 \in G$, $h, h_1 \in H$.

In particular, as the conjugation action of a group G on itself satisfies (1), then the *tensor square* $G \otimes G$ of a group G may always be defined.

In [4] Brown and Loday presented a topological significance for the non-abelian tensor product. They showed that the third homotopy group of the suspension of an Eilenberg-MacLane space $K(G, 1)$ satisfies

$$\pi_3 SK(G, 1) \cong J_2(G),$$

where $J_2(G)$ denotes the kernel of the derived map $\kappa : G \otimes G \to G'$, $g \otimes h \overset{\kappa}{\mapsto} [g, h] = g^{-1}h^{-1}gh$. Also, the non-abelian tensor product is used to describe the third relative homotopy group of a triad as a (non-abelian) tensor product of the second homotopy groups of appropriate subspaces. More specifically, let a CW-complex X be the union $X = A \cup B$ of two path-connected CW-subspaces A and B whose intersection $C = A \cap B$ is path-connected. Suppose that the canonical homomorphisms $\pi_1(C) \to \pi_1(A)$, $\pi_1(C) \to \pi_1(B)$ are surjective. Then, according to [3],

$$\pi_3(X, A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C),$$

where the groups $\pi_2(A, C)$ and $\pi_2(B, C)$ act on one another via $\pi_1(C)$.

So, computing $G \otimes H$ has some topological interest besides its relevance as an intrinsic group theoretical problem.

In [15,2,7] it is presented the concept of a tensor product modulo q, where q is a non-negative integer, which generalizes the concept of non-abelian tensor product and has connections with homology groups, universal q-central extensions and q-capability of groups.

The purpose of this note is to report on a group construction in connection with non-abelian tensor products of groups and recent development in non-abelian tensor products and q-tensor products. The proofs of the results are omitted in the interest of brevity.

We mention the survey paper by Kappe [19] which contains an account on the progress in non-abelian tensor products from 1987 up to 1997. Also, Morse [28] gives a survey on the computation of the non-abelian tensor square of groups. In this work, we attempt to minimize overlap with these two surveys.

Notation in this survey is fairly standard. For elements x, y, z in a group G, the conjugate of x by y is $x^y = y^{-1}xy$; and the commutator of x and y is $[x, y] = x^{-1}x^y$. As usual we write G' for the derived subgroup of G, G^{ab} for the abelianized group G/G', $\mathrm{d}(G)$ for the minimal number of generators for G and $\exp(G)$ for the exponent of G.

2. The commutator approach

The investigation of the non-abelian tensor product from a group theoretical point of view started with a paper by Brown, Johnson, and Robertson [5]. In that work the authors compute the non-abelian tensor square of all non-abelian groups of order up to 30 using Tietze transformations. However, this method is not appropriate for computing $G \otimes G$ for larger groups since we have $|G|^2$ generators and $2|G|^3$ relations. Thus it is interesting to look for more effective methods to computing $G \otimes G$.

We observe that the defining relations of the tensor product can be viewed as abstractions of commutator relations; thus in [32] it is considered the following related construction: Let G and H be groups and $\varphi : H \to H^\varphi$ an isomorphism (H^φ is an isomorphic copy of H, where $h \mapsto h^\varphi$, for all $h \in H$). Define the group $\nu(G)$ to be

$$\nu(G) := \langle G, G^\varphi \mid [g_1, {g_2}^\varphi]^{g_3} = [{g_1}^{g_3}, ({g_2}^{g_3})^\varphi] = [g_1, {g_2}^\varphi]^{{g_3}^\varphi}, \ \ \forall g_1, g_2, g_3 \in G\rangle.$$

Independently, Ellis and Leonard [13] studied a similar construction.

The motivation for studying $\nu(G)$ is the commutator connection:

Proposition 2.1 *([32, Proposition 2.6]) The map $\Phi : G \otimes G \to [G, G^\varphi]$, defined by $g \otimes h \mapsto [g, h^\varphi]$, $\forall g, h \in G$, is an isomorphism.*

This isomorphism is useful to compute the non-abelian tensor square of G inside $\nu(G)$. We observe that simplified presentations for $\nu(G)$ can be obtained from certain generating sequences of G associated to some subnormal series (see [33], [23]) to compute a concrete representation of $\nu(G)$. Having computed such a representation of $\nu(G)$, the non-abelian tensor square is then obtained from that as the subgroup $[G, G^\varphi]$.

Many structural aspects of $\nu(G)$ relative to G have been investigated so far, which help in computing $\nu(G)$. For instance

Theorem 2.1 *([32, Proposition 2.4, Theorem A])*

(i) If G is a finite π-group, π a set of primes, then $\nu(G)$ is a finite π-group;

(ii) If G is nilpotent of class c, then $\nu(G)$ is nilpotent of class at most $c + 1$;

(iii) If G is solvable of derived length d, then $\nu(G)$ is solvable of derived length at most $d + 1$.

Hence a solvable or nilpotent quotient algorithm can be used to compute $\nu(G)$ whenever G is solvable or nilpotent. Use of this strategy was made to perform computations using different computer algebra systems: for example, Ellis&Leonard [13] used CAYLEY; Ellis [12] used Magma; McDermott [23] and Rocco [33] used GAP [38].

Theorem 2.2 *([33, Theorem 2.1]) Let G be a finite solvable group given by a power-conjugate presentation with polycyclic generating sequence $S = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ and S-relations R (respectively S^φ-relations R^φ; see (16) for details). Then* *(i)* $\nu(G)$ *has the presentation*

$$\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{a}_1^{\varphi}, \ldots, \mathfrak{a}_n^{\varphi} \mid R, R^{\varphi}, [\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}]^{\mathfrak{a}_k} = [\mathfrak{a}_i{}^{\mathfrak{a}_k}, (\mathfrak{a}_j{}^{\mathfrak{a}_k})^{\varphi}] = [\mathfrak{a}_i, \mathfrak{a}_j]^{\mathfrak{a}_k^{\varphi}}, \quad 1 \le i, j, k \le n \rangle.$$

(ii) The subgroup $[G, G^{\varphi}]$ *is generated by the set* $\{[\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}], 1 \le i, j \le n\}$.

These results were used with the GAP system [38] to compute $\nu(G)$, $G \otimes G$ and other invariants of G, for all non-abelian p-groups G of order up to $p^4, p = 2,\ 3$ (see [33], Table I). McDermott [23] extended these results end those of Ellis and Leonard [13] by using generating sets associated to other subnormal series of G to write a number of GAP routines for computing tensor products.

Blyth and Morse extended results in [33] to compute $G \otimes G$ for polycyclic groups G. Their results for polycyclic groups are summarized in the following theorem.

Theorem 2.3 *([1, Theorem 4]) Let* G *be a polycyclic group given by a finite presentation* $\langle \mathcal{G} | \mathcal{R} \rangle$ *and polycyclic generating set* $\mathfrak{G}$. *Then*

(i) $\nu(G)$ *and* $[G, G^{\varphi}]$ *are polycyclic.*

(ii) $\nu(G)$ *has a presentation that depends only on* $\mathcal{G}$, $\mathcal{R}$ *and* $\mathfrak{G}$.

(iii) The subgroup $[G, G^{\varphi}]$ *is generated by the set* $\{[g^{\pm 1}, h^{\pm \varphi}] | g, h \in \mathfrak{G}\}$.

Eick and Nickel [9] gave a polycyclic presentation of $\nu(G)$ for G polycyclic and an algorithm to compute the Schur multiplier and the non-abelian tensor square of a polycyclic group.

3. $G \otimes H$ and the group $\eta(G, H)$

A construction related to the non-abelian tensor product was introduced by Ellis and Leonard [13] and played a significant role in their computation of non-abelian tensor product of finite groups. In [29] it was defined as follows: Let G and H be groups acting compatibly on each other and H^{φ} an extra copy of H, isomorphic through $\varphi : H \to H^{\varphi},\ h \mapsto h^{\varphi}$, for all $h \in H$. The construction is defined to be the group

$$\eta(G, H) = \langle G, H^{\varphi} \mid \quad [g, h^{\varphi}]^{g_1} = [g^{g_1}, (h^{g_1})^{\varphi}],\ [g, h^{\varphi}]^{h_1^{\varphi}} = [g^{h_1}, (h^{h_1})^{\varphi}], \quad \forall g, g_1 \in G,\ h, h_1 \in H \rangle.$$

Note that $\eta(G, H)$ is isomorphic with the group $G * H/J$ of [13]. When $G = H$ and all actions are conjugations, $\eta(G, H)$ becomes the group $\nu(G)$.

It follows from Proposition 1.4 in [16] that there is an isomorphism from the subgroup $[G, H^{\varphi}]$ of $\eta(G, H)$ onto the non-abelian tensor product $G \otimes H$, such that $[g, h^{\varphi}] \mapsto g \otimes h$, for all $g \in G$ and $h \in H$. We observe that $[G, H^{\varphi}]$ is a normal subgroup of $\eta(G, H)$ and that $\eta(G, H) = ([G, H^{\varphi}] \cdot G) \cdot H^{\varphi}$, where the dots denote semidirect products.

One of the themes of research on the non-abelian tensor products has been to determine which group properties are preserved by non-abelian tensor products. By using homological arguments, Ellis [10] showed that if G and H are finite groups, then $G \otimes H$ is also finite. Recently, Thomas [35] presented a purely group theoretic proof of the result by Ellis. In [37,29] it is studied solvability and nilpotency of $G \otimes H$. In [29] it is also given a description of the lower central series and of the derived series of $G \otimes H$. More precisely, let $\gamma_i(G)$ (resp. G_i) denote the ith term of the lower central series (resp. derived series) of an arbitrary group G. If H acts on G, we write $[G,H]$ for the subgroup $\langle g^{-1}g^h \mid g \in G, h \in H\rangle$ of G. Then

Theorem 3.1 *([29, Theorem A])*

(i) For all $i \geq 2$, $\gamma_i(G \otimes H)$ is isomorphic to the subgroup $[\gamma_{i-1}([G,H]), [G,H]^{\varphi}]$ of $\eta(G,H)$.

(ii) For all $i \geq 1$, $(G \otimes H)_i$ is isomorphic to $[[G,H]_{i-1}, [H,G]^{\varphi}_{i-1}]$ of $\eta(G,H)$.

As a consequence, if $[G,H]$ is nilpotent (resp. solvable), then $G \otimes H$ is nilpotent (resp. solvable). Moravec [25] showed that the tensor product of polycyclic groups is also polycyclic, generalizing part (i) of Theorem 2.3. Furthermore, he established

Theorem 3.2 *([26, Theorem 1]) Let M and N be locally finite groups acting compatibly upon each other. Then the group $M \otimes N$ is locally finite. If furthermore M and N have finite exponents that are π-numbers, then $exp(M \otimes N)$ is also a π-number and can be bounded by a function depending only on $exp(M)$ and $exp(N)$.*

In the case when M and N are two normal subgroups of a group G, the group $M \otimes N$ can be replaced with the group $\eta(M,N)$ in Theorem 3.2 (see [26], Corollary 5).

Write $\eta^*(A,H)$ to denote the group $\eta(A,H)$ when A is an abelian H-group acting trivially on H. If B is any H-subgroup of A, then $B \cdot H$ means the semidirect product of B by H. Besides the embedding of $G \otimes H$ into $\eta(G,H)$, certain split extensions can also be embedded into $\eta(G,H)$.

Proposition 3.1 *([31, Propositions A,B]) Let A and H be as above*

(i) If A and H are finite and $(|A|, |H|) = 1$, then $[A, H] \cdot H$ is embedded into $\eta^*(A,H)$.

(ii) If A is finite and there is a central element $h \in H$ such that h acts fixed-point-free on A, then $A \cdot H$ is embedded into $\eta^*(A,H)$.

Recall that a finite group G containing a proper subgroup $H \neq 1$ such that $H \cap H^g = 1$ for all $g \in G \setminus H$ is called a *Frobenius group*. The subgroup H is called a *Frobenius complement*. By a celebrated theorem of Frobenius, the set $N = G \setminus (\cup_{x \in G}(H^*)^x)$ is a normal subgroup of G (called its *Frobenius kernel*) such that $G = NH$ and $N \cap H = 1$. We have $|H|$ divides $|N|-1$. If $|H| = |N|-1$, then we say that G is a *complete Frobenius group*; here the kernel N is an elementary abelian p-group for some prime p (see for instance [36] for an overview). As a consequence of the above result, we have

Proposition 3.2 *([31])*

(i) Every Frobenius group with an abelian Kernel A and complement H is embedded into $\eta^*(A, H)$;

(ii) If F denotes the finite field with q elements $GF(q)$, then the affine group $\mathcal{A}_n(F)$ is embedded into $\eta^*(A, \mathrm{GL}_n(F))$, where $A \cong (F^n, +)$ is the translation subgroup.

4. Some bounds for $|G \otimes G|$

For a finite p-group G of order p^n and $|G'| = p^m$, it is proved in [32] that $|G \otimes G|$ divides $p^{n(n-m)}$. Later, McDermott [23], using the orders of the factors of the lower central series of G, established a bound for $|G \otimes G|$ which improves the above bound. He showed that if G is a $d-$generator $p-$group of order $|G| = p^n$, then $|G \otimes G|$ divides p^{nd}. This result was extended by Ellis and McDermott [14] to $G \otimes H$, where G and H are prime-power groups. Recently, Moravec [27] found a new bound which improves the previous estimates.

Theorem 4.1 *([27, Theorem 3.6]) Let G be a finite p-group of exponent p^e. If $r = max\{\mathrm{d}(H) \ : \ H \leq G\}$, then set $m = \lceil \log_2 r \rceil$ if $p > 2$ and $m = \lceil \log_2 r \rceil + 1$ otherwise. Then $|G \otimes G| \leq p^{r^2(2e+m)}$.*

Let us denote by $I(H)$ the augmentation ideal of $\mathbb{Z}[H] \to \mathbb{Z}$. By considering groups $\eta(A, B)$ with appropriate arguments A and B, the first named author established in [29] a bound for the order of the non-abelian tensor square of a finite solvable group G involving the terms of the derived series of G.

Theorem 4.2 *([29, Theorems B, 3.3])*

(i) If G is a finite solvable group of derived length l, then

$$|G \otimes G| \leq \quad |G^{ab} \otimes_{\mathbb{Z}} G^{ab}| \prod_{i=1}^{l-1} (|G_i^{ab} \otimes_{\mathbb{Z}} G_i^{ab}|^{2^{i-1}} \cdot |G_i^{ab} \otimes_{\mathbb{Z}[\frac{G}{G_i}]} I(\tfrac{G}{G_i})|) \cdot \\ \cdot \prod_{i=1}^{l-2} \prod_{k=i-1}^{l-1} |G_k^{ab} \otimes_{\mathbb{Z}[\frac{G_i}{G_k}]} I(\tfrac{G_i}{G_k})|^{2^{i-1}}.$$

(ii) If G is a finite metabelian group (i.e. $l = 2$), then

$$|G \otimes G| \text{ divides } |G^{ab} \otimes_{\mathbb{Z}} G^{ab}||G' \wedge G'||G' \otimes_{\mathbb{Z}[G^{ab}]} I(G^{ab})|.$$

In the particular case of a finite metabelian group G where $|G'|$ and $|G^{ab}|$ are coprime we have the precise order of $G \otimes G$ in terms of $|G'|$, $|G^{ab}|$ and the order of the G^{ab}-stable subgroup of the Schur multiplier $\mathrm{M}(G')$ (see [20] for an overview):

Theorem 4.3 *([31, Proposition C, Corollary 4])* *(i) Assume G is a metabelian group as above. Then $|G \otimes G| = n|G'| \cdot |G^{ab} \otimes G^{ab}|$, where n is the order of the G^{ab}-stable subgroup of* $\mathrm{M}(G')$.

(ii) If in addition $\mathrm{M}(G') = 1$ *then* $G \otimes G \cong G' \times (G^{ab} \otimes G^{ab})$.

5. q-Tensor product

Let G and H be normal subgroups of some group L and q a non-negative integer. The definition of the *q-tensor product*, $G \otimes^q H$, of G and H has evolved in papers [15,2,7]. Let $\widehat{\mathcal{K}} = \{\widehat{k}\}$ be a set of symbols, one for each $k \in G \cap H$ (If $q = 0$ then $\widehat{\mathcal{K}}$ is taken to be the empty set). According to [11], we may define $G \otimes^q H$ as the group generated by the symbols $g \otimes h$ and $\widehat{k}$, for $g \in G$, $h \in H$ and $\widehat{k} \in \widehat{\mathcal{K}}$, subject to the following relations (for all $g, g_1 \in G$, $h, h_1 \in H$ and $k, k_1 \in G \cap H$):

$$g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}); \tag{2}$$

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h); \tag{3}$$

$$(g \otimes h)^{\widehat{k}} = g^{(k^q)} \otimes h^{(k^q)}; \tag{4}$$

$$\widehat{kk_1} = \widehat{k} \prod_{i=1}^{q-1} (k \otimes (k_1^{-i})^{k^{q-1-i}}) \widehat{k_1}; \tag{5}$$

$$[\widehat{k}, \widehat{k_1}] = k^q \otimes k_1^q; \tag{6}$$

$$\widehat{[g,h]} = (g \otimes h)^q. \tag{7}$$

For $q = 0$ the 0-tensor product $G \otimes^0 H$ is the non-abelian tensor product $G \otimes H$. In the particular case where $G = H = L$ it is called the q-tensor square, $G \otimes^q G$, of G.

The *q-exterior product* of G and H, denoted by $G \wedge^q H$, is defined to be the quotient of $G \otimes^q H$ by its (normal) subgroup generated by the elements $k \otimes k$, for all $k \in G \cap H$. We write $g \wedge h$ to denote the image in $G \wedge^q H$ of the generator $g \otimes h$.

Let $\rho' : G \wedge^q H \to H$ be the homomorphism induced by the map $\rho : G \otimes^q H \to H$, $g \otimes h \mapsto [g, h], \widehat{k} \mapsto k^q$, for all $g \in G$, $h \in H$ and $k \in G \cap H$. We denote by $H_n(G, \mathbb{Z}_q)$ the n-th homology group of G with coefficients in the trivial G-module $\mathbb{Z}_q$. Ellis and Rodríguez-Fernández [15] established the following relations between homology groups of G and q-exterior products.

Theorem 5.1 *([15, Corollary 2]) Let G be any group. Then*

(i) $H_2(G, \mathbb{Z}_q) \cong \mathrm{Ker}(\rho' : G \wedge^q G \to G)$.

(ii) For any free presentation F/R of G, $H_3(G, \mathbb{Z}_q) \cong \mathrm{Ker}(\rho' : R \wedge^q F \to F)$.

The q-exterior square is also related with universal q-central extensions. A *q-central extension* is a central extension $1 \to Z \to E \to G \to 1$ such that every element of Z has order dividing q. We say that this extension is *universal* if for any other q-central extension $1 \to Z' \to E' \to G \to 1$, there is a unique morphism of extensions

$$\begin{array}{ccccccccc} 1 & \to & Z & \to & E & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow 1 & & \\ 1 & \to & Z' & \to & E' & \to & G & \to & 1. \end{array}$$

The existence and the structure of universal q-central extensions were studied by Brown in [2] (see also Conduché and Rodríguez-Fernández [7]). By using Theorem 5.1, Brown proved that if G is a q-perfect group, that is, $G = G'G^q$, where G^q is the subgroup of G generated by the set $\{g^q | g \in G\}$, then universal q-central extensions of G are isomorphic to the sequence

$$1 \to H_2(G, \mathbb{Z}_q) \to G \wedge^q G \xrightarrow{\rho'} G \to 1.$$

We remind that the *q-centre* of a group G is the subgroup $Z_q(G)$ of the center $Z(G)$ consisting of those elements with order dividing q. The group G is said to be *q-capable* if there exists a group Q such that $Z(Q) = Z_q(Q)$ and $G \cong Q/Z(Q)$. The following central subgroup of G, called the *q-exterior center* of G, was considered in [11]

$$Z_q^{\wedge}(G) = \{g \in G \,|\, 1 = g \wedge x \in G \wedge^q G, \ \forall x \in G\}.$$

This subgroup is useful in deciding whether G is q-capable or not, according to the next result.

Theorem 5.2 *([11, Proposition 16 (vii)]) A group G is q-capable if and only if its q-exterior centre is trivial.*

Given normal subgroups G and H of some group L, denote by $G\sharp^q H$ the subgroup of L generated by commutators $[g, h]$ and q-th powers k^q for $g \in G$, $h \in H$ and $k \in G \cap H$. In the following theorem, we compile some properties of q-tensor products and q-exterior products found in [7,11].

Theorem 5.3 *Suppose that G and H are normal subgroups of a group L and let q be a non-negative integer.*

(i) ([7]) For $r \geq 1$ there is an exact sequence

$$G \otimes^{qr} H \xrightarrow{\phi} G \otimes^r H \to G \cap H / G\sharp^q H \to 1,$$

where the homomorphism ϕ is defined by $g \otimes h \mapsto g \otimes h$ and $\widehat{k} \mapsto \widehat{k^q}$ for all $g \in G$, $h \in H$ and $k \in G \cap H$;

(ii) ([7]) If $G \cap H = G\sharp^q H$, then $G \otimes^q H \cong G \wedge^q H$;

(iii) ([7]) If $[G, H] = 1$, then $G \otimes^q H \cong (G/G\sharp^q G) \otimes_{\mathbb{Z}} (H/H\sharp^q H)$;

(iv) ([11]) If F/R is a free presentation of G, then $G \wedge^q G \cong F'F^q/R^q[R, F]$.

6. The group $\eta^q(G,H)$

A construction related to the q-tensor product was introduced by Ellis in [11]. Using a slightly different approach, in [6] it was defined in the following manner: Let G and H be normal subgroups of some larger group L and suppose that all actions are given by conjugation in L. As in Section 5, for $q \geq 1$ put $\mathcal{K} = G \cap H$ and let $\widehat{\mathcal{K}} = \{\widehat{k} \,|\, k \in \mathcal{K}\}$ be a set of symbols, one for each element of $\mathcal{K}$ (for $q = 0$, $\widehat{\mathcal{K}}$ is defined to be the empty set). Let $F(\widehat{\mathcal{K}})$ be the free group over $\widehat{\mathcal{K}}$ and $\eta(G,H) * F(\widehat{\mathcal{K}})$ be the free product of $\eta(G,H)$ and $F(\widehat{\mathcal{K}})$. As G and H^φ are embedded into $\eta(G,H)$, the elements of G (respectively of H^φ) can be identified with their respective images in $\eta(G,H) * F(\widehat{\mathcal{G}})$. Let J denote the normal closure in $\eta(G,H) * F(\widehat{\mathcal{K}})$ of the following elements, for all $\widehat{k}, \widehat{k_1} \in \widehat{\mathcal{K}},\ g \in G$ and $h \in H$:

$$g^{-1}\, \widehat{k}\, g\, (\widehat{k^g})^{-1}; \tag{8}$$

$$(h^\varphi)^{-1}\, \widehat{k}\, h^\varphi\, (\widehat{k^h})^{-1}; \tag{9}$$

$$(\widehat{k})^{-1}[g,h^\varphi]\, \widehat{k}\, [g^{k^q}, (h^{k^q})^\varphi]^{-1}; \tag{10}$$

$$(\widehat{k})^{-1}\, \widehat{kk_1}\, (\widehat{k_1})^{-1}(\prod_{i=1}^{q-1}[k,(k_1^{-i})^\varphi]^{k^{q-1-i}})^{-1}; \tag{11}$$

$$[\widehat{k}, \widehat{k_1}]\, [k^q, (k_1^q)^\varphi]^{-1}; \tag{12}$$

$$\widehat{[g,h]}\, [g,h^\varphi]^{-q}. \tag{13}$$

The construction is defined to be the factor group

$$\eta^q(G,H) := (\eta(G,H) * F(\widehat{\mathcal{K}}))/J.$$

It is clear that if R denotes the set of the relators corresponding to (8), ..., (13) and $S = \{[g,h^\varphi]^{g_1}[g^{g_1},(h^{g_1})^\varphi]^{-1},\ [g,h^\varphi]^{h_1^\varphi}[g^{h_1},(h^{h_1})^\varphi]^{-1} \mid g,g_1 \in G,\ h,h_1 \in H\}$, then the group $\eta^q(G,H)$ has the presentation:

$$\eta^q(G,H) = \langle G, H^\varphi, \widehat{\mathcal{K}} \mid S, R\rangle. \tag{14}$$

Also, if $q = 0$, then $\eta^0(G,H)$ coincides with the group $\eta(G,H)$.

According to [6] the elements $g \in G$ and $h^\varphi \in H^\varphi$ can be identified with their respective images in $\eta^q(G,H)$. Let K denote the subgroup of $\eta^q(G,H)$ generated by the images of $\widehat{\mathcal{K}}$. The relators (10) imply that K normalizes the subgroup $[G,H^\varphi]$ in $\eta^q(G,H)$ and hence $\Upsilon^q(G,H) = [G,H^\varphi]K$ is a normal subgroup of $\eta^q(G,H)$. It follows from [11, Theorem 8] (see also [6]) that $\Upsilon^q(G,H)$ is isomorphic to $G \otimes^q H$, for any $q \geq 0$. Furthermore, $\eta^q(G,H) = H^\varphi \cdot (G \cdot \Upsilon^q(G,H))$.

When $G = H = L$ we write $\nu^q(G)$ and $\Upsilon^q(G)$ to denote $\eta^q(G,G)$ and $\Upsilon^q(G,G)$, respectively. We remark that the properties for $\nu(G)$ given by Theorem 2.1 and part (i) of Theorem 2.3 also hold for $\nu^q(G)$, according to [6, Theorem 2.8].

Now let $G \neq \{1\}$ be a polycyclic group. By [34] such group has a consistent power-conjugate presentation, that is, G has a consistent presentation in the form

$$G = \langle \mathfrak{a}_1, \ldots, \mathfrak{a}_n \mid \mathfrak{pcr}(G)\rangle, \tag{15}$$

where $\mathfrak{pcr}(G)$ is the set of the following power-conjugate relations:

$$\begin{aligned} \mathfrak{a}_j^{\mathfrak{a}_i} &= \mathfrak{a}_{i+1}^{e(i,j,i+1)} \cdots \mathfrak{a}_n^{e(i,j,n)}, \text{ for } 1 \le i < j \le n, \\ \mathfrak{a}_j^{\mathfrak{a}_i^{-1}} &= \mathfrak{a}_{i+1}^{f(i,j,i+1)} \cdots \mathfrak{a}_n^{f(i,j,n)}, \text{ for } 1 \le i < j \le n, \\ \mathfrak{a}_i^{r_i} &= \mathfrak{a}_{i+1}^{l(i,i+1)} \cdots \mathfrak{a}_n^{l(i,n)}, \text{ for } i \in I. \end{aligned} \tag{16}$$

In [6] it is showed that for a polycyclic group G, the defining relations of the group $\nu^q(G)$ can be defined only in the polycyclic generators of G, with the exception of the relation (11).

Theorem 6.1 *([6, Corollaries 3.5, 3.6]) Let G be a polycyclic group given by a consistent power-conjugate presentation as in (15), and $\widehat{\mathcal{G}} = \{\widehat{g}\,|g \in G\}$. Then*

(i) $\nu^q(G)$ has the presentation

$$\begin{aligned} \nu^q(G) = \langle & \mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{a}_1^{\varphi}, \ldots, \mathfrak{a}_n^{\varphi}, \widehat{\mathcal{G}} \,|\, \mathfrak{pcr}(G), \mathfrak{pcr}(G^{\varphi}), \\ & [\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}]^{\mathfrak{a}_k^{\gamma}} = [\mathfrak{a}_i^{\mathfrak{a}_k^{\gamma}}, (\mathfrak{a}_j^{\mathfrak{a}_k^{\gamma}})^{\varphi}] = [\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}]^{(\mathfrak{a}_k^{\gamma})^{\varphi}}, \ \widehat{\mathfrak{a}_i}^{\mathfrak{a}_k^{\gamma}} = \widehat{(\mathfrak{a}_i^{\mathfrak{a}_k^{\gamma}})}, \\ & \widehat{\mathfrak{a}_i}^{(\mathfrak{a}_k^{\gamma})^{\varphi}} = \widehat{(\mathfrak{a}_i^{\mathfrak{a}_k^{\gamma}})}, [\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}]^{\widehat{\mathfrak{a}_k^{\gamma}}} = [\mathfrak{a}_i^{\mathfrak{a}_k^{\gamma q}}, (\mathfrak{a}_i^{\mathfrak{a}_k^{\gamma q}})^{\varphi}], \\ & \widehat{gh} = \widehat{g}\,(\prod_{i=1}^{q-1} [g, (h^{\varphi})^{-i}]^{g^{q-1-i}})\,\widehat{h}, \ \ [\widehat{\mathfrak{a}_i^{\alpha}}, \widehat{\mathfrak{a}_j^{\beta}}] = [\mathfrak{a}_i^{\alpha q}, (\mathfrak{a}_j^{\beta q})^{\varphi}], \\ & \widehat{[\mathfrak{a}_i^{\alpha}, \mathfrak{a}_j^{\beta}]} = [\mathfrak{a}_i^{\alpha}, (\mathfrak{a}_j^{\beta})^{\varphi}]^q, \ 1 \le i, j, k \le n, \ \forall\, \widehat{gh}, \widehat{g}, \widehat{h} \in \widehat{\mathcal{G}}\,\rangle, \ \textit{where} \end{aligned}$$

$$\alpha = \begin{cases} 1 & \textit{if } \ o(\mathfrak{a}_i) < \infty, \\ \pm 1 & \textit{if } \ o(\mathfrak{a}_i) = \infty, \end{cases} \ \beta = \begin{cases} 1 & \textit{if } \ o(\mathfrak{a}_j) < \infty, \\ \pm 1 & \textit{if } \ o(\mathfrak{a}_j) = \infty, \end{cases} \ \gamma = \begin{cases} 1 & \textit{if } \ o(\mathfrak{a}_k) < \infty, \\ \pm 1 & \textit{if } \ o(\mathfrak{a}_k) = \infty; \end{cases}$$

(ii) $\Upsilon^q(G)$ is generated by

$$\{[\mathfrak{a}_i, \mathfrak{a}_i^{\varphi}], [\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}][\mathfrak{a}_j^{\varphi}, \mathfrak{a}_i], [\mathfrak{a}_i^{\alpha}, (\mathfrak{a}_j^{\varphi})^{\beta}], \widehat{\mathfrak{a}_k}, \textit{for } \ 1 \le i < j \le n, \ 1 \le k \le n\},$$

$$\textit{where } \alpha = \begin{cases} 1 & \textit{if } \ o(\mathfrak{a}_i) < \infty, \\ \pm 1 & \textit{if } \ o(\mathfrak{a}_i) = \infty, \end{cases} \ \textit{and } \beta = \begin{cases} 1 & \textit{if } \ o(\mathfrak{a}_j) < \infty, \\ \pm 1 & \textit{if } \ o(\mathfrak{a}_j) = \infty; \end{cases}$$

(iii) The subgroup $\Delta^q(G) = \langle [g, g^{\varphi}] | g \in G\rangle$ of $\nu^q(G)$ is generated by the set

$$\{[\mathfrak{a}_i, \mathfrak{a}_i^{\varphi}], [\mathfrak{a}_i, \mathfrak{a}_j^{\varphi}][\mathfrak{a}_j^{\varphi}, \mathfrak{a}_i], \ \textit{for } 1 \le i < j \le n\}.$$

For G a polycyclic group given by a pc-presentation, Theorem 6.1 considerably reduces the number of generators and relators of $\nu^q(G)$ when compared with the presentation (14). Furthermore, when G is finite it is useful to perform efficient computer computations of $\nu^q(G)$ and of its subgroups $\Upsilon^q(G)$ and $\Delta^q(G)$ (and, consequently, of the $q-$exterior square of G since $G \wedge^q G \cong \Upsilon^q(G)/\Delta^q(G)$). In fact, with the help of the GAP system [38], in [6] it is computed $|\nu^q(G)|$, $G \otimes^q G$ and $G \wedge^q G$ for the non-abelian groups G of orders up to 14 and $q \in \{2, 3, 4, 5\}$.

Recently, Martins [22] extended results in [9] and described an algorithm for computing a polycyclic presentation of the q-multiplier, the second homology group with coefficients in $\mathbb{Z}_q$, the q-exterior square and the q-tensor square of a polycyclic group given by a polycyclic presentation, for any $q \geq 0$.

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