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Finiteness of Hermitian Levels of Some Algebras

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ABSTRACT: We characterize the hermitian levels of quaternion and octonion algebras and of an 8-dimensional algebra $\mathcal{D}$ over the ground field F, constructed using a weak version of the Cayley-Dickson double process.

It is shown that all values of the hermitian levels of quaternion algebras with the hat-involution also occur as hermitian levels of $\mathcal{D}$. We give some limits to the levels of the algebra $\mathcal{D}$ over some ground field.

Key Words: Quaternion algebra, octonion algebra, level, hermitian level, sums of squares, quadratic forms.

Contents

1. Introduction

Lewis [2] define hermitian level of a ring with identity equipped with non-trivial involution. He showed that there exist commutative rings with involutions having any positive integer as hermitian level. He also showed that the hermitian levels of quadratic extensions of fields and the quaternion division algebras, both with standard involutions, are power of two, if they are finite. In this case, any power of two can occur as hermitian level. For a quaternion division algebra equipped with standard and hat-involution, Lewis has obtained results relating the finiteness of the hermitian level with certain sum of squares in the ground field of the algebra. These results were extended to octonion algebras by Pumplün and Unger in [4].

In this paper, we obtain characterizations about finiteness of hermitian levels of quaternion (respectively; octonion) algebras $\mathcal{D}$, with involutions in terms of the hermitian length of one element $b \in F$, or length of $-b$ in a quadratic extension field $F(\sqrt{a})$ (respectively; biquadratic extension field $F(\sqrt{c}, \sqrt{-a})$), where F is the ground field of $\mathcal{D}$, and the elements $a,\ b,\ c \in F$ are related with the generators of $\mathcal{D}$, (Propositions 3.2, 3.3, 3.4).

Based on the weak version of the Cayley-Dickson doubling process, we use the hat-involution over a quaternion algebra $\mathcal{Q}$ and obtain a new algebra, which we

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denote by $\mathcal{Q}(\sqrt{c})$, equipped with a non-trivial involution which is not scalar. We show that the finiteness of the hermitian level of this algebras and of the octonion algebra has similar behavior (Propositions 3.3 and 3.5). We show that all values of hermitian levels of quaternion algebra occur as hermitian levels of the algebra $\mathcal{Q}(\sqrt{c})$, (Proposition (4.1)). In addition, we give some limits to the level of the algebra $\mathcal{Q}(\sqrt{c})$ over some ground field.

2. Preliminaries

Let $\mathcal{D}$ be a ring with an identity (and $2 \neq 0$) equipped with a non-trivial involution, i.e., an anti-automorphism $* : \mathcal{D} \to \mathcal{D}$ of period two. We denote by $n \times \langle 1 \rangle$, the hermitian form over $(\mathcal{D}, *)$ represented by the $n \times n$ identity matrix, i.e., the form $h : \mathcal{D}^n \times \mathcal{D}^n \to \mathcal{D}$ given by $h\big((x_1, \ldots, x_n), (y_1 \ldots, y_n)\big) = \sum_{r=1}^{n} x_r^* y_r$. The subset of elements of $\dot{\mathcal{D}}$ $(:= \mathcal{D} \backslash \{0\})$ represented by h is $D_H n \times \langle 1 \rangle := \{\sum_{r=1}^{n} x_r^* x_r \in \dot{\mathcal{D}},\ x_r \in \mathcal{D}\}$. Also, for a quadratic form over a field F, $q : V \to F$, (where V is a n-dimensional vectorspace over F), $D_Q q := \{q(u) \in \dot{F},\ u \in V\}$ is the subset of elements of F represented by q. The quadratic form q is said to be *isotropic* if there exists $u \in V$, $u \neq 0$ such that $q(u) = 0$. The vector u is said to be *isotropic*. In otherwise q and u are said to be *anisotropic*. The sum of n quadratic form $q :\quad q \perp q \perp \cdots \perp q$ (n times) is denoted by $n \times q$. In particular $n \times \langle 1 \rangle$ also denotes the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$. If q_1 and q_2 are isometric quadratic forms, we will denote this fact by $q_1 \sim q_2$. The quadratic form $\langle a \rangle \otimes q$ will be denoted by $a.q$. An n-fold Pfister form over a field F is a quadratic form of the type $\langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$, also denoted by $\langle\langle a_1, a_2, \ldots, a_n \rangle\rangle$. It is known that if q is an n-fold Pfister form over a field F, then $a.q \sim q$, for every $a \in D_Q q$. In particular $D_Q q$ is a subgroup of $\dot{F}$. These and other basic results on quadratic and hermitian forms can be found in [1] and [5].

Definition 2.1 Let $\mathcal{D}$ be a ring equipped with an involution $*$. An *hermitian square* in $\mathcal{D}$, (or in $(\mathcal{D}, *)$), is an element of the form $x^* x$ for some $x \in \dot{\mathcal{D}}$, i.e., an element of $D_H \langle 1 \rangle$. The *hermitian length* of $c \in \dot{\mathcal{D}}$ is the least (positive) integer n such that c is a sum of n hermitian squares in $\mathcal{D}$, i.e., $c \in D_H n \times \langle 1 \rangle$. In otherwise the *hermitian length* is infinite. We denote by $l(\mathcal{D}, *, c)$ the hermitian length of c, and by $l(\mathcal{D}, c)$ if $*$ is the identity map of $\mathcal{D}$, i.e., $l(\mathcal{D}, c)$ is the usual *length* of $c \in \mathcal{D}$.

The *hermitian level* of $(\mathcal{D}, *)$ is the hermitian length of $-1 \in \mathcal{D}$, and it is denoted by $s(\mathcal{D}, *)$.

The Cayley-Dickson doubling process

Now, we consider a finite dimensional F-algebra A equipped with a *scalar involution* $*$, that is, the trace of $x\colon x^* + x$ and the norm of $x\colon x^* x$ belong to F.

The *Cayley-Dickson doubling process* is a well-known way to construct new algebra with scalar involution from a given algebra with scalar involution. If A is an F-algebra with involution $*$ and $c \in \dot{F}$, then the F-vectorspace $A \times A$ becomes an F-algebra $\mathcal{D}$ with the multiplication

$$(u,v)(u_1,v_1) = (uu_1 + cv_1^* v, v_1 u + vu_1^*),$$

for $u,\ u_1,\ v,\ v_1 \in A$, with involution

$$(u,v)^* = (u^*, -v).$$

This new algebra is denoted by $Cay(A,c)$, and we say that is obtained from A by the Cayley-Dickson doubling process.

Since A is F-isomorphic to $A \times \{0\}$, if we denote $(0,1)$ by e, we can denote the algebra $A \times A$ by $A \oplus Ae$. Thus, the multiplication in $A \oplus Ae$ is

$$(u+ve)(u_1+v_1e) = \big(uu_1 + cv_1^* v\big) + \big(v_1 u + vu_1^*\big)e, \qquad (*)$$

and the involution is: $(u+ve)^* = u^* - ve$.

The involution $*$ is a scalar involution on $A \oplus Ae$ if and only if $*$ is a scalar involution on A. In this case, $N_{A\oplus Ae}(u+ve) = N_A(u) - cN_A(v)$.

If $*$ is the identity map on F and $a \in \dot{F}\backslash F^2$, $Cay(F,a)$ is usually denoted by $F(i)$ ($\simeq F(\sqrt{a})$). $F(i)$ is an F-algebra equipped with the standard involution $\overline{\alpha + \beta i} = \alpha - \beta i$, and norm form $N := \langle 1, -a\rangle$. If $x = \alpha + \beta i \in F(i)$, then the hermitian square $\overline{x}x$ is $N(\alpha,\beta) = \alpha^2 - a\beta^2$. The hermitian form $n \times \langle 1\rangle$ has $n \times N$ as underlying quadratic form. Thus, $s(\mathcal{D},-) = n$, if and only if, $-1 \in D_H n \times \langle 1\rangle = D_Q n \times N$, but $-1 \notin D_Q(n-1) \times N$. In this case, we write

$$-1 = \sum_{r=1}^{n} N(\alpha_r, \beta_r) \quad \text{or} \quad -1 = \sum_{r=1}^{n} (\alpha_r^2 - a\beta_r^2), \qquad (1)$$

where $x_r = \alpha_r + \beta_r\sqrt{a} \in \mathcal{D}$.

If $A = F(i),\ i^2 = a \in \dot{F}$, then $Cay(A,b) = A \oplus Aj$ is a quaternion algebra $\mathcal{Q}$ over F, denoted by $\big((a,b)/F\big)$. As an F-vectorspace, $\mathcal{Q}$ has the canonical basis $\{1,i,j,k\}$, where $ij = -ji = k$, and $j^2 = b$. In particular $\mathcal{Q}$ is noncommutative. The standard involution in $\mathcal{Q}$ is $\overline{x+yj} = \overline{x} - yj$, and the norm form of $\mathcal{Q}$ is $N = \langle\langle -a,-b\rangle\rangle$. For $u = x + yj = \alpha + \beta i + \gamma j + \delta k \in \mathcal{Q}$, the hermitian square $\overline{u}u$ is $N(\alpha,\beta,\gamma,\delta) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2$: an element of F. Thus $s(\mathcal{Q},-) = n$, if and only if, $-1 \in D_Q n \times N$ and $-1 \notin D_Q(n-1) \times N$. We write

$$-1 = \sum_{r=1}^{n} (\alpha_r^2 - a\beta_r^2 - b\gamma^2 + ab\delta^2) \quad \text{or} \quad -1 = \sum_{r=1}^{n} N(u_r), \qquad (2)$$

where $u_r = \alpha_r + \beta_r i + \gamma_r j + \delta_r k \in \mathcal{Q}$.

In the third case, we consider an octonion algebra $\mathcal{O}$ obtained from the quaternion algebra $\mathcal{Q}$ and $c \in \dot{F}$ by Cayley-Dickson doubling process. If $\mathcal{Q} = \big((a,b)/F\big)$, then the octonion algebra $Cay(\mathcal{Q},c)$ is also denoted by $\mathcal{O} := \big((a,b,c)/F\big)$. Since

$\mathcal{Q}$ is noncommutative and $i(je) \neq (ij)e$ we see that $\mathcal{O}$ is noncommutative and nonassociative algebra. For more details, see ([3], §2) and ([4], §2.2).

The standard involution on $\mathcal{O}$ is given by $\overline{u+ve} = \overline{u} - ve$, for every $u,\ v \in \mathcal{Q}$. This involution is scalar and the norm form of $\mathcal{O}$ is $N_{\mathcal{O}}(u+ve) = N_{\mathcal{Q}}(u) - cN_{\mathcal{Q}}(v)$. Since $N_{\mathcal{Q}} = \langle\langle -a, -b\rangle\rangle$ we have $N := N_{\mathcal{O}} = \langle\langle -a, -b, -c\rangle\rangle$. Thus, if $u + ve = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k + \alpha_5 e + \alpha_6 ie + \alpha_7 je + \alpha_8 ke \in \mathcal{O}$, then the hermitian square $(\overline{u+ve})(u+ve) = N(\alpha_1, \ldots, \alpha_8) \in F$. Consequently, if $s(\mathcal{O}, -)$ is finite, then $s(\mathcal{O}, -)$ is the least positive integer n such that $-1 \in D_Q n \times N$, that is, $-1 = \sum_{r=1}^{n} N(\alpha_{1r}, \ldots, \alpha_{8r})$. We write

$$-1 = \sum_{r=1}^{n} (\alpha_{1r}^2 - a\alpha_{2r}^2 - b\alpha_{3r}^2 + ab\alpha_{4r}^2 - c\alpha_{5r}^2 + ac\alpha_{6r}^2 + bc\alpha_{7r}^2 - abc\alpha_{8r}^2). \tag{3}$$

Lewis [2] has defined and studied the hat-involution $\widehat{\ }$ on quaternion algebras and Pumplüm and Unger [4], have generalized the hat-involution to octonion algebras.

If $u = \alpha + \beta i + \gamma j + \delta k$ then $\widehat{u}$ is defined by $\widehat{u} = \alpha - \beta i + \gamma j + \delta k$. Thus, the hermitian square $\widehat{u}u$ is the following element in $\mathcal{Q}$: $\widehat{u}u = \alpha^2 - a\beta^2 + b\gamma^2 - ab\delta^2 + 2(\alpha\gamma - a\beta\delta)j + 2(\alpha\delta - \beta\gamma)k$. Therefore, if $s(\mathcal{Q}, \widehat{\ })$ is finite, then $s(\mathcal{Q}, \widehat{\ })$ is the smallest integer n such that $-1 = \sum_{r=1}^{n} \widehat{u}_r u_r$ for $u_1,\ u_2, \ldots, u_n$ in $\mathcal{Q}$, that is, the following system of quadratic forms is satisfied

$$\left.\begin{aligned} \sum_{r=1}^{n} \alpha_r^2 - a\sum_{r=1}^{n} \beta_r^2 + b\sum_{r=1}^{n} \gamma_r^2 - ab\sum_{r=1}^{n} \delta_r^2 = -1, \\ \text{and} \quad \sum_{r=1}^{n} (\alpha_r\gamma_r - a\beta_r\delta_r) = \sum_{r=1}^{n} (\alpha_r\delta_r - \beta_r\gamma_r) = 0. \end{aligned}\right\} \tag{4}$$

In the last case we consider the following weak version of the Cayley-Dickson doubling process: From the hat-involution on $\mathcal{Q}$ and (*) we get:

$$(u+ve)(u_1+v_1e) = (uu_1 + c\widehat{v_1}v) + (v_1u + v\widehat{u_1})e, \tag{$**$}$$

where $u, u_1, v, v_1 \in \mathcal{Q}$. This yields a new algebra and it is easy to prove that $\widehat{u+ve} = \widehat{u} + ve$ is an involution on this new algebra. From now on, we denote this algebra by $\mathcal{Q}(\sqrt{c})$.

An hermitian square in $(\mathcal{Q}(\sqrt{c}), \widehat{\ })$ is $(\widehat{u+ve})(u+ve) = (\widehat{u}u + c\widehat{v}v) + 2v\widehat{u}e$. If $u = x_1 + x_2 i + x_3 j + x_4 k$ and $v = y_1 + y_2 i + y_3 j + y_4 k$ then
$\widehat{u}u = (x_1^2 - ax_2^2 + bx_3^2 - abx_4^2) + 2(x_1x_3 - ax_2x_4)j + 2(x_1x_4 - x_2x_3)k$ and $v\widehat{u} = (x_1y_1 - ax_2y_2 + bx_3y_3 - abx_4y_4) + (x_1y_2 - x_2y_1 + bx_3y_4 - bx_4y_3)i + (x_3y_1 + x_1y_3 + ax_4y_2 + ax_2y_4)j + (x_4y_1 + x_1y_4 + x_3y_2 + x_2y_3)k$. Thus $s(\mathcal{Q}(\sqrt{c}), \widehat{\ }) = n$ (finite) if,

and only if, n is the least integer such that the following equations are satisfied

$$\left.\begin{aligned}
\sum_{r=1}^{n}\left(x_{1r}^2-ax_{2r}^2+bx_{3r}^2-abx_{4r}^2+cy_{1r}^2-acy_{2r}^2+bcy_{3r}^2-abcy_{4r}^2\right)&=-1,\\
\sum_{r=1}^{n}(x_{1r}x_{3r}-ax_{2r}x_{4r}+cy_{1r}y_{3r}-acy_{2r}y_{4r})&=0,\\
\sum_{r=1}^{n}(x_{1r}x_{4r}-x_{2r}x_{3r}+cy_{1r}y_{4r}-cy_{2r}y_{3r})&=0,\\
\sum_{r=1}^{n}(x_{1r}y_{1r}-ax_{2r}y_{2r}+bx_{3r}y_{3r}-abx_{4r}y_{4r})&=0,\\
\sum_{r=1}^{n}(x_{1r}y_{2r}-x_{2r}y_{1r}+bx_{3r}y_{4r}-bx_{4r}y_{3r})&=0,\\
\sum_{r=1}^{n}(x_{3r}y_{1r}+x_{1r}y_{3r}+ax_{4r}y_{2r}+ax_{2r}y_{4r})&=0 \text{ and}\\
\sum_{r=1}^{n}(x_{4r}y_{1r}+x_{1r}y_{4r}+x_{3r}y_{2r}+x_{2r}y_{3r})&=0.
\end{aligned}\right\} \tag{5}$$

Since $\mathcal{D}$ can be noncommutative and nonassociative ring, we need of the following set

$Assoc(\mathcal{D},*)$ =: $\{B\in\mathcal{D}$ such that B is invertible and $\big((B^{-1})^*(X^*X)\big)B^{-1} = (XB^{-1})^*(XB^{-1})$, for every $X\in\mathcal{D}\}$.

If $-$ is the standard involution on $\mathcal{O}$, then $Assoc(\mathcal{O},-)=\{B\in\mathcal{O}\mid B$ is invertible$\}$, because $\overline{X}X=N(X)\in F$ for every $X\in\mathcal{O}$, and N is multiplicative. In particular if $\mathcal{O}$ is a division algebra then $Assoc(\mathcal{O},-)=\dot{\mathcal{O}}$.

Definition 2.2 The hermitian form $n\times\langle 1\rangle$ over $(\mathcal{D},*)$ is *isotropic* if there exist $x_1,x_2,\dots,\ x_n\in\mathcal{D}$, at least one $x_i\in Assoc(\mathcal{D},*)$, such that $\sum_{i=1}^n x_i^*x=0$.

Lemma 2.1 Let $\mathcal{Q}(\sqrt{c})$ be the algebra given as earlier with multiplication $(**)$. If $\mathcal{Q}=\big((a,b)/F\big)$ is a quaternion division algebra, then $Assoc(\mathcal{Q}(\sqrt{c}),\widehat{\ })=\dot{F}\cup\dot{F}i\cup\dot{F}e$. In particular $Assoc(\mathcal{Q}(\sqrt{c}),\widehat{\ })$ is non-empty.

Proof: Since $\mathcal{Q}(\sqrt{c})$ is distributive, it suffices to show that $\big(\widehat{A}(\widehat{X}Y)\big)A=(\widehat{XA})(YA)$ for $X,\ Y\in\{1,i,j,k,e,ie,je,ke\}$ if, and only if, $A\in F\cup Fi\cup Fe$. A straightforward calculation shows it. ∎

The next two lemmas have the same proof of the ([2], Lemma 1.1 and Lemma 1.2) and ([4], Corollary 3.3 and Lemma 3.4).

Lemma 2.2 Let $\mathcal{D}$ a ring equipped with a non-trivial involution $*$ such that $Assoc(\mathcal{D},*)$ is non-empty. Then $s\big(\mathcal{D},*\big)=n$, if and only if, the hermitian form $n\times\langle 1\rangle$ is anisotropic but the hermitian form $(n+1)\times\langle 1\rangle$ is isotropic.

Lemma 2.3 Let $\mathcal{D}$ a ring equipped with a non-trivial involution $*$ such that $Assoc(\mathcal{D}, *)$ is non-empty. The following statements are equivalent:

(i) Zero is a non-trivial sum of hermitian squares.

(ii) -1 is a sum of hermitian squares.

(iii) Each symmetric element of $\mathcal{D}$, that is, $x \in \mathcal{D} \mid x^* = x$, is a sum of hermitian squares.

3. Main Results

The following propositions give relations between levels and lengths of some elements.

Proposition 3.1 Let $\mathcal{D} = F(\sqrt{a})$ be a quadratic extension of the field F equipped with the standard involution $-$ and $N = \langle 1, -a\rangle$. Then

(i) $s(\mathcal{D}, -) \leq \min\{s(F),\ l(F, a)\}$ and for every $c \in D_Q N$, $l(\mathcal{D}, -, -c) = s(\mathcal{D}, -)$. In particular $l(\mathcal{D}, -, a) = s(\mathcal{D}, -)$.

(ii) If $s(\mathcal{D}, -) = n$ and $l(F, -a) = m$, then $s(F) \leq n(m + 1)$.

(iii) $s(\mathcal{D}, -)$ is finite, if and only if, $l(F, a)$ is finite.

Proof: Using the equation (1) the proof is analogous to the proof of the next proposition. ∎

Remark 3.1 We observe that, if $\mathcal{Q} = \big((a, b)/F\big)$ and $\mathcal{O} = \big((a, b, c)/F\big)$ are not division algebras, then $s\big(\mathcal{Q}, -\big) = s\big(\mathcal{O}, -\big) = 1$. Indeed, by ([1], Chapter III, Proposition 2.4 and Chapter X, Theorem 1.7) the norm forms $\langle\langle -a, -b\rangle\rangle \sim 2 \times \langle 1, -1\rangle$ and $\langle\langle -a, -b, -c\rangle\rangle \sim 4 \times \langle 1, -1\rangle$, and so $s\big(\mathcal{Q}, -\big) = s\big(\mathcal{O}, -\big) = 1$. So, when we deal with the levels of the algebras $\mathcal{Q}$ and $\mathcal{O}$ equipped with the standard involutions, it is suffices to consider division algebras. But, this is not true for quaternion algebra and the algebra $\mathcal{Q}(\sqrt{c})$ equipped both with involution $\widehat{\ }$. For instance, take $\mathcal{Q}_1 = \big((-2, 3)/\mathbb{Q}\big)$ and $\mathcal{Q}_1(\sqrt{3})$. From equations (4) and (5), it is clear that $s(\mathcal{Q}_1, \widehat{\ }) = s(\mathcal{Q}_1(\sqrt{3}), \widehat{\ }) = \infty$, but $\mathcal{Q}_1$ and $\mathcal{Q}_1(\sqrt{3})$ are not division algebras, because $(1 - i - j)(1 + i + j) = 0$.

Take $\mathcal{Q}_2 = \big((-1, -3)/\mathbb{Q}\big)$ and $\mathcal{Q}_2(\sqrt{-3})$. Then $\mathcal{Q}_2$ is a division algebra, but $\mathcal{Q}_2(\sqrt{-3})$ is not a division algebra with the multiplication given by (**), because $(-j + e)(j + e) = 0$. We will prove in the example 3.1(b) that $s(\mathcal{Q}_2(\sqrt{-3}), \widehat{\ }) = 2$.

Proposition 3.2 Let $\mathcal{Q} = \big((a, b)/F\big)$ be a quaternion division algebra equipped with the standard involution $-$ and $N = \langle\langle -a, -b\rangle\rangle$. Then

(i) $s(\mathcal{Q}, -) \leq \min\{s(F),\ l(F, a), l(F, b), l(F, -ab)\}$ and for every $d \in D_Q N$, $l(\mathcal{Q}, -, -d) = s(\mathcal{Q}, -)$. In particular $l(\mathcal{Q}, -, a) = l(\mathcal{Q}, -, b) = s(\mathcal{Q}, -)$.

(ii) If $s(\mathcal{Q}, -) = n$ and $m = \max\{l(F, -a), l(F, -b), l(F, ab)\}$ then $s(F) \leq n(3m + 1)$.

(iii) The following statements are equivalent

(a) $s(\mathcal{Q}, -)$ is finite.
(b) b is a sum of hermitian squares in $F(\sqrt{a})$, i.e., $l(F(\sqrt{a}), -, b)$ is finite.
(c) a is a sum of hermitian squares in $F(\sqrt{b})$, i.e., $l(F(\sqrt{b}), -, a)$ is finite.

Proof: (i) If $s(F)$, $l(F,a), l(F,b)$ and $l(F,-ab)$ are infinite, then it is obvious that $s(\mathcal{Q},-) \leq \min\{s(F),\ l(F,a), l(F,b), l(F,-ab)\}$. If $s(F) = n$: finite, then $-1 = \sum_{r=1}^{n} \alpha_r^2$ $(\alpha_r \in \dot{F})$ and equation (2) is satisfied. If $l(F,a) = n$ (finite) then $a = \sum_{r=1}^{n} \beta_r^2$, $\beta_r \in \dot{F}$. Thus $-1 = -a \sum_{r=1}^{n} \left(\beta_r/a\right)^2$, and so the equation (2) is satisfied. Therefore $s(\mathcal{Q},-) \leq l(F,a)$. Similarly we have $s(\mathcal{Q},-) \leq l(F,b)$ and $s(\mathcal{Q},-) \leq l(F,-ab)$.

If $d \in D_Q N$, then by ([1], Chapter X, Theorem 1.8) $d.N \sim N$, and therefore $D_Q d.N = D_Q N$. Thus, if $-1 \in D_H n \times \langle 1 \rangle$, then $-1 = \sum_{r=1}^{n} N(\alpha_r, \beta_r, \gamma_r, \delta_r)$ for some $\alpha_r, \beta_r, \gamma_r, \delta_r \in F$. Multiplying both sides of this equality by d we get $-d = \sum_{r=1}^{n} d.N(\alpha_r, \beta_r, \gamma_r, \delta_r) = \sum_{r=1}^{n} N(\alpha'_r, \beta'_r, \gamma'_r, \delta'_r)$. Then $-d \in D_H n \times \langle 1 \rangle$, and so $l(\mathcal{Q}, -, -d) \leq s(\mathcal{Q}, -)$. Suppose now $-d = \sum_{r=1}^{n} N(\alpha_r, \beta_r, \gamma_r, \delta_r)$. Multiplying both sides of this equality by d, we obtain $-d^2 = \sum_{r=1}^{n} d.N(\alpha_r, \beta_r, \gamma_r, \delta_r)$, or $-d^2 = \sum_{r=1}^{n} N(\alpha'_r, \beta'_r, \gamma'_r, \delta'_r)$, because $D_Q d.N = D_Q N$. It follows that $-1 = \sum_{r=1}^{n} N\left(\alpha'_r/d, \beta'_r/d, \gamma'_r/d, \delta'_r/d\right)$. Consequently $s(\mathcal{Q},-) \leq l(\mathcal{Q},-,-d)$, and we conclude that $s(\mathcal{Q},-) = l(\mathcal{Q},-,-d)$.

(ii) Take representations of $-a$, $-b$ and ab as sums of m squares in F. Replacing $-a$, $-b$ and ab in the equation (2) by these sums, it yields a representation of -1 as a sum of $n + 3mn$ squares in F. Thus $s(F) \leq n(3m+1)$.

(iii) (a) $\Longrightarrow$ (b) We will consider $\mathcal{Q}_K = \mathcal{Q} \otimes_F K \simeq \big((a,b)/K\big)$ where $K = F(\sqrt{-a})$. Since $-a$ is a square in K, and $s(\mathcal{Q}_K, -)$ is finite, from the equation (2) we obtain $bS_1^2 = (1+S)S_1$ and so b is a sum of squares in K. Write $b = \sum_{r=1}^{m} (x_r + y_r\sqrt{-a})^2$. Then $b = \sum_{r=1}^{m} x_r^2 - a \sum_{r=1}^{m} y_r^2$, (and $\sum_{r=1}^{m} x_r y_r = 0$) in F. Thus b is sum of hermitian squares in $F(\sqrt{a})$.

(b) $\Longrightarrow$ (a) If $l\big(F(\sqrt{a}), -, b\big)$ is finite then $b = \sum_{r=1}^{m} \gamma_r^2 - a \sum_{r=1}^{m} \delta_r^2$. It follows that $-1 = -b \sum_{r=1}^{m} \left(\gamma_r/b\right)^2 + ab \sum_{r=1}^{m} \left(\delta_r/b\right)^2$, and so the equation (2) is satisfied.

(a) $\Longleftrightarrow$ (c) is similar to (a) $\Longleftrightarrow$ (b). ∎

Proposition 3.3 Let $\mathcal{O} = \big((a,b,c)/F\big)$ be an octonion division algebra equipped with the standard involution $-$ and $\mathcal{Q} = \big((a,b)/F\big)$. Then

(i) $s(\mathcal{O},-) \leq \min\{s(F), l(F,a), l(F,b), l(F,c), l(F,-ab), l(F,-ac), l(F,-bc), l(F,abc)\}$ and for every $d \in D_Q N$, $l(\mathcal{O},-,-d) = s(\mathcal{O},-)$. In particular $l(\mathcal{O},-,a) = l(\mathcal{O},-,b) = l(\mathcal{O},-,c) = s(\mathcal{O},-)$.

(ii) If $s(\mathcal{O},-) = n$ and $m = \max\{l(F,-a), l(F,-b), l(F,-c), l(F,ab), l(F,ac), l(F,bc), l(F,-abc)\}$ then $s(F) \leq n.(7m+1)$.

(iii) If $K = F(\sqrt{-c})$, then the following conditions are equivalent
(a) $s(\mathcal{O},-)$ is finite,
(b) b is a sum of hermitian squares in $K(\sqrt{a})$, i.e., $l(K(\sqrt{a}), -, b)$ is finite.
(c) a is a sum of hermitian squares in $K(\sqrt{b})$, i.e., $l\big(K(\sqrt{b}), -, a\big)$ is finite.

Proof: The proofs of (i) and (ii) are similar to the proof of (i) and (ii) of the Proposition 3.2, respectively.

(iii) (a) $\Longrightarrow$ (b) We will take $\mathcal{O}_K =: \mathcal{O} \otimes_F K \simeq \big((a,b,c)/K\big)$, where $K = F\big(\sqrt{-c}\big)$. Then $s(\mathcal{O}_K, -)$ is finite and so the equation (3) is true over K. Replacing $-c$ by $\sqrt{-c}^2$ in equation (3), we have $-1 = S_1 - aS_2 - bS_3 + abS_4$, where $S_i,\ i = 1,2,3,4$ are each sums of $2n$ squares in K. Then $s((a,b)/K,-)$ is finite. From Proposition 3.2(iii) this implication is completed.

(b) $\Longrightarrow$ (a) If $l\,(K(\sqrt{a}),-,b)$ is finite, then $b = \sum_{r=1}^m x_r^2 - a\sum_{r=1}^m y_r^2$, x_r, $y_r \in K$. Let $x_r = z_r + t_r\sqrt{-c}$, $y_r = l_r + p_r\sqrt{-c}$ be in K. Then $b = \sum_{r=1}^m z_r^2 - c\sum_{r=1}^m t_r^2 - a\sum_{r=1}^m l_r^2 + ac\sum_{r=1}^m p_r^2$, and $\sum_{r=1}^m (z_rt_r - al_rp_r) = 0$, in F. From the first equation, it follows that $-1 = -b\sum_{r=1}^m \big(z_r/b\big)^2 + ab\sum_{r=1}^m \big(l_r/b\big)^2 + bc\sum_{r=1}^m \big(t_r/b\big)^2 - abc\sum_{r=1}^m \big(p_r/b\big)^2$ and the equation (3) is satisfied, and so $s\big(\mathcal{O},-\big)$ is finite. This completes the equivalence of (a) and (b).

The proof of (a) $\Longleftrightarrow$ (c) is similar and the proof of the proposition is finished. ∎

The following propositions show similar results for the involution $\widehat{\ }$.

Proposition 3.4 Let $\mathcal{Q} = \big((a,b)/F\big)$ be a quaternion algebra equipped with the involution $\widehat{\ }$ given as earlier. Then

(i) $s(\mathcal{Q}, \widehat{\ }) \leq \min\{s(F), l(F,a), l(F,-b), l(F,ab)\}$.

(ii) If $s(\mathcal{Q}, \widehat{\ }) = n$ and $m = \max\{l(F,-a), l(F,b), l(F,-ab)\}$ then $s(F) \leq n(3m+1)$.

(iii) The following statements are equivalent

(a) $s(\mathcal{Q}, \widehat{\ })$ is finite,

(b) $-b$ is a sum of squares in $F(\sqrt{-a})$, i.e., $l\big(F(\sqrt{-a}),-b\big)$ is finite.

(c) a is a sum of squares in $F(\sqrt{b})$, i.e., $l\big(F(\sqrt{b}),a\big)$ is finite.

(d) $-b$ is a sum of hermitian squares in $F(\sqrt{a})$, i.e., $l\big(F(\sqrt{a}),-,-b\big)$ is finite.

Proof: The proofs of (i) and (ii) are similar to proof of (i) and (ii) of the Proposition 3.2, respectively. For instance, if $l(F,-b) = n$, then $-b = \sum_{r=1}^n {\gamma'_r}^2$. It yields $-1 = b\sum_{r=1}^n \big(\gamma'_r/b\big)^2$. If we take $\gamma_r = \gamma'_r/b$, $\alpha_r = \beta_r = \delta_r = 0$, for every $r = 1,\ldots,r$, we see that the equations in (4) are all satisfied. Thus $s(\mathcal{Q}, \widehat{\ }) \leq n$.

(iii) (a) $\Longrightarrow$ (b) We will consider $\mathcal{Q}_K = \mathcal{Q} \otimes_F K \simeq \big((a,b)/K\big)$, where $K = F(\sqrt{-a})$. Then $s(\mathcal{Q}_K, \widehat{\ })$ is finite and so the equation (4) is true over K. Since $-a$ is a square in K, by equation (4) we obtain $-1 = S_1 + bS_2$, where $S_1,\ S_2$ are both sums of squares in K. Hence $-bS_2^2 = (1+S_1)S_2$ and so $-b$ is a sum of squares in K.

(b) $\Longrightarrow$ (a) Let $-b = \sum_{r=1}^m (\gamma'_r + \delta'_r\sqrt{-a})^2$. Then $-b = \sum_{r=1}^m \gamma_r'^2 - a\sum_{r=1}^m \delta_r'^2$ and $\sum_{r=1}^m \gamma'_r\delta'_r = 0$ in F. Thus $-1 = b\sum_{r=1}^m \gamma_r^2 - ab\sum_{r=1}^m \delta_r^2$ and $\sum_{r=1}^m \gamma_r\delta_r = 0$, where $\gamma_r = \gamma'_r/b$ and $\delta_r = \delta'_r/b$. Putting $\alpha_r = \beta_r = 0$ for all $r = 1,\ldots,m$, we see that the equations in (4) are all satisfied. Thus $s(\mathcal{Q}, \widehat{\ })$ is finite.

(a) $\Longleftrightarrow$ (c) is similar to (a) $\Longleftrightarrow$ (b) replacing $-a$ by b and $-b$ by a.

(b) $\Longrightarrow$ (d) By hypothesis $-b = \sum_{r=1}^m (x_r + y_r\sqrt{-a})^2$ in $F(\sqrt{-a})$. Thus $-b = \sum_{r=1}^m x_r^2 - a\sum_{r=1}^m y_r^2$ (and $\sum_{r=1}^m x_ry_r = 0$) in F, and so $s\big(F(\sqrt{a}),-,-b\big)$ is finite.

(d) $\Longrightarrow$ (a) If $-b = \sum_{r=1}^{m} x_r^2 - a\sum_{r=1}^{m} y_r^2$, then $-1 = b\sum_{r=1}^{m}(x_r/b)^2 - ab\sum_{r=1}^{m}(y_r/b)^2$. For $\alpha_r = \beta_r = 0$, $\gamma_r = x_r/b$ and $\delta_r = y_r/b$, $r = 1, 2, \ldots, m$ we see that the equations in (4) are all satisfied, and so $s(\mathcal{Q}, \widehat{\ })$ is finite. ∎

Proposition 3.5 Let $\mathcal{Q} = \big((a,b)/F\big)$ be a quaternion algebra, $\mathcal{Q}(\sqrt{c})$ be the algebra with multiplication give as (**). Then

(i) $s(\mathcal{Q}(\sqrt{c}), \widehat{\ }) \leq \min\{s(F),\ l(F,a),\ l(F,-b),\ l(F,-c),\ l(F,ab),\ l(F,ac),\ l(F,-bc),\ l(F,abc)\}$.

(ii) If $s(\mathcal{Q}(\sqrt{c}), \widehat{\ }) = n$ and $m = \max\{l(F,-a),\ l(F,b),\ l(F,c),\ l(F,-ab),\ l(F,-ac),\ l(F,bc),\ l(F,-abc)\}$ then $s(F) \leq n.(7m+1)$.

(iii) If $K = F(\sqrt{c})$, then the following conditions are equivalent

(a) $s(\mathcal{Q}(\sqrt{c}), \widehat{\ })$ is finite,

(b) $-b$ is a sum of squares in $K(\sqrt{-a})$, i.e., $l(K(\sqrt{-a}), -b)$ is finite,

(c) a is a sum of squares in $K(\sqrt{b})$, i.e., $l\big(K(\sqrt{b}), a\big)$ is finite,

(d) $-b$ is a sum of hermitian squares in $K(\sqrt{a})$, i.e., $l\big(K(\sqrt{a}), -, -b\big)$ is finite.

Proof: The proof of (i) and (ii) are similar to proof of (i) and (ii) of the Proposition 3.2, respectively.

(iii) Suppose that (a) is true. By hypothesis the equation (5) is satisfied. Replacing y_{ir} by $x_{ir} = \sqrt{c}y_{ir}$ in the first, second and third equations of (5) we get the equations of (4) over $K = F(\sqrt{c})$. Thus $s\big(\big((a,b)/K\big), \widehat{\ }\big)$ is finite. From Proposition 3.4 it follows that (b), (c) and (d) are true and equivalent.

Conversely, if (c) is true (equivalently, (b) and (d) are true, by Proposition 3.4 (iii)), then $a = \sum_{r=1}^{n} u_r^2$, where $u_r = z_r + t_r\sqrt{b}$ in $K(\sqrt{b})$. Thus $a = \sum_{r=1}^{n}(z_r^2 + bt_r^2)$ and $\sum_{r=1}^{n} z_r t_r = 0$ in $K = F(\sqrt{c})$. Let $z_r = x_r' + y_r'\sqrt{c}$, $t_r = x_r'' + y_r''\sqrt{c}$ be in $F(\sqrt{c})$. Replacing z_r and t_r in the equations above, we obtain $a = \sum_{r=1}^{n}({x_r'}^2 + c{y_r'}^2 + b{x_r''}^2 + bc{y_r''}^2)$, $\sum_{r=1}^{n}(x_r'y_r' + bx_r''y_r'') = 0$, $\sum_{r=1}^{n}(x_r'x_r'' + cy_r'y_r'') = 0$, and $\sum_{r=1}^{n}(x_r'y_r'' + x_r''y_r') = 0$. These equations yield

$$\sum_{r=1}^{n}\Big(-a\big(\tfrac{x_r'}{a}\big)^2 - ab\big(\tfrac{x_r''}{a}\big)^2 - ac\big(\tfrac{y_r'}{a}\big)^2 - abc\big(\tfrac{y_r''}{a}\big)^2\Big) = -1,$$

$$\sum_{r=1}^{n}\Big(\tfrac{x_r'}{a}.\tfrac{y_r'}{a} + b\tfrac{x_r''}{a}.\tfrac{y_r''}{a}\Big) = 0, \qquad \sum_{r=1}^{n}\Big(\tfrac{x_r'}{a}.\tfrac{y_r''}{a} + \tfrac{x_r''}{a}.\tfrac{y_r'}{a}\Big) = 0, \text{ and}$$

$$\sum_{r=1}^{n}\Big(\tfrac{x_r'}{a}.\tfrac{x_r''}{a} + c\tfrac{y_r'}{a}.\tfrac{y_r''}{a}\Big) = 0.$$

For $x_{1r} = x_{3r} = y_{1r} = y_{3r} = 0$, $x_{2r} = x_r'/a$, $x_{4r} = x_r''/a$, $y_{2r} = y_r'/a$ and $y_{4r} = y_r''/a$, we see that the equations in (5) are all satisfied. It proves that $s(\mathcal{Q}(\sqrt{c}), \widehat{\ })$ is finite and the proposition is finished. ∎

Example 3.1 **(a)** Let $\mathcal{Q}_1 = \big((-1,3)/\mathbb{Q}\big)$ and $\mathcal{Q}_2 = \big((-1,-3)/\mathbb{Q}\big)$. Then $s(\mathcal{Q}_1, \widehat{\ }) = \infty$, because $D_Q n \times \langle 1,1,3,3\rangle \subset P$ and $-1 \notin P$. Also, it is easy to see that $l(\mathbb{Q},-3) = l\big(\mathbb{Q}(\sqrt{3}), -1\big) = l\big(\mathbb{Q}(\sqrt{-1}), -, -3\big) = \infty$. Thus $\mathcal{Q}_1$ satisfies the Proposition 3.4.

The example of ([2], p. 474), shows that $s(\mathcal{Q}_2, \widehat{\ })=2$. From the Hasse-Minkowski Principle ([1], VI, 3.1) it follows that $l(\mathbb{Q},3)=3$ and $l\big(\mathbb{Q}(\sqrt{-3}),-1\big)\leq 3$. It is clear that $l(\mathbb{Q}(i),-,3)=2$. Therefore $\mathcal{Q}_2$ satisfies the Proposition 3.4.

(b) Now, let us show that $s\big(\mathcal{Q}_1(\sqrt{3}),\widehat{\ }\big)=\infty$ and $s\big(\mathcal{Q}_2(\sqrt{-3}),\widehat{\ }\big)=2$.

Let $K=\mathbb{Q}(\sqrt{3})$. Since -3 is not a sum of squares in K, by Proposition 3.5(iii) it follows that $s(\mathcal{Q}_1(\sqrt{3}),\widehat{\ })=\infty$. Now, suppose $s(\mathcal{Q}_2(\sqrt{-3}),\widehat{\ })=1$. Then $-1=\widehat{q}q$, for some $q\in\mathcal{Q}_2(\sqrt{-3})$. Write $q=u+ve$, where $u,\ v\in\mathcal{Q}_2=\big((-1,-3)/\mathbb{Q}\big)$. Then $-1=\widehat{q}q=(\widehat{u}+ve)(u+ve)$ implies that

$$-1=(\widehat{u}u-3\widehat{v}v)+2v\widehat{u}e.$$

Thus $-1=\widehat{u}u-3\widehat{v}v$ and $v\widehat{u}=0$ in $\mathcal{Q}_2$. Since $\mathcal{Q}_2$ is a division algebra, either $u=0$ or $v=0$. Now $v=0$ implies $-1=\widehat{u}u$ in $\mathcal{Q}_2$ which is impossible, because $s\big(\mathcal{Q}_2,\widehat{\ }\big)=2$. Also $u=0$ implies $-1=-3\widehat{v}v$, that is, $3=(\widehat{3v})(3v)$. Write $3v=x+yj$, for $x,y\in\mathbb{Q}(i)$. Replacing in the equation above we get $3=(\widehat{x}x-3y\widehat{y})+2\widehat{x}yj$. Thus $3=\widehat{x}x-3y\widehat{y}$ and $\widehat{x}y=0$ in $\mathbb{Q}(i)$. It follows that $x=0$ or $y=0$. If $y=0$, then $3=\widehat{x}x=\alpha^2+\beta^2$, (where $x=\alpha+\beta i\in\mathbb{Q}(i)$), which is insoluble in $\mathbb{Q}$. If $x=0$, then $3=-3y\widehat{y}$ or $-1=y\widehat{y}$. Let $y=r+si$. Then $-1=r^2+s^2$ in $\mathbb{Q}$, absurd. Thus $s(\mathcal{Q}_2(\sqrt{-3}),\widehat{\ })\geq 2$. Since $s(\mathcal{Q}_2(\sqrt{-3})),\widehat{\ })\leq s(\mathcal{Q}_2,\widehat{\ })=2$ (because $\mathcal{Q}_2\subset\mathcal{Q}_2(\sqrt{-3})$), we have $s(\mathcal{Q}_2(\sqrt{-3}),\widehat{\ })=2$.

4. Some Levels and Some Bounds

Remark 4.1 For the next propositions we will consider the seven homogeneous equations obtained from equation (5) replacing the first equation by $\sum_{r=1}^{n}\big(x_{1r}^2-ax_{2r}^2+bx_{3r}^2-abx_{4r}^2+cy_{1r}^2-acy_{2r}^2+bcy_{3r}^2-abcy_{4r}^2\big)=0$.

Similarly we have three homogeneous equations obtained from equation (4), and two homogeneous equations obtained from equations (2) and (3).

Proposition 4.1 Let $\mathcal{Q}=\big((a,b)/F\big)$ and $\mathcal{Q}_1=\big((a,b)/F(t)\big)$ be quaternion division algebras and $\mathcal{O}=\big((a,b,t)/F(t)\big)$, where t is one indeterminate over F. Then $s(\mathcal{Q}_1(\sqrt{t}),\widehat{\ })=s(\mathcal{Q},\widehat{\ })$ and $s(\mathcal{O},-)=s(\mathcal{Q},-)$.

Proof: Since $\mathcal{Q}\subset\mathcal{Q}_1(\sqrt{t})$ we have $s(\mathcal{Q}_1(\sqrt{t}),\widehat{\ })\leq s(\mathcal{Q},\widehat{\ })$. Suppose that $s(\mathcal{Q}_1(\sqrt{t}),\widehat{\ })=m$. From Lemmas 2.1, 2.2 and Remark (4.1), it follows that there exists a non-trivial solution in $F(t)$ of

$$\left.\begin{aligned}\sum_{r=1}^{m+1}\big(x_{1r}^2-ax_{2r}^2+bx_{3r}^2-abx_{4r}^2\big)+t\sum_{r=1}^{m+1}\big(y_{1r}^2-ay_{2r}^2+by_{3r}^2-aby_{4r}^4\big)=0,\\ \sum_{r=1}^{m+1}\big(x_{1r}x_{3r}-ax_{2r}x_{4r}\big)+t\sum_{r=1}^{m+1}\big(y_{1r}y_{3r}-ay_{2r}y_{4r}\big)=0,\\ \sum_{r=1}^{m+1}\big(x_{1r}x_{4r}-x_{2r}x_{3r}\big)+t\sum_{r=1}^{m+1}\big(y_{1r}y_{4r}-y_{2r}y_{3r}\big)=0,\quad\cdots\end{aligned}\right\}\qquad(6)$$

obtained from equation (5). Write $x_{ir} = f_{ir}/g_{ir}$, $y_{ir} = h_{ir}/l_{ir} \in F(t)$, with $g_{ir} = 1$ ($l_{ir} = 1$), if $x_{ir} = 0$ (respect. $y_{ir} = 0$). Take $g = \prod_{r=1}^{m+1}\left(\prod_{i=1}^{4} g_{ir}\right)$, $l = \prod_{r=1}^{m+1}\left(\prod_{i=1}^{4} l_{ir}\right)$. Multiplying the equations in (6) by $(gl)^2$, it yields a non-trivial solution in $F[t]$ of equation (6). Thus we can assume that x_{ir} and y_{ir} are polynomials in t. Dividing by an convenient power of t, if necessary, we may assume that one of the x_{ir} has a non-zero constant term. (If the constant terms of all x_{ir} are equal to zero, then dividing by t we obtain a similar equation interchanging x_{ir} by y_{ir}). Putting $t = 0$ we have one non-trivial solution of the homogeneous equations of (4), according to Remark 4.1. Since $Assoc(\mathcal{Q}, \hat{})$ is non-empty, by Lemma 2.2, we get $s(\mathcal{Q}, \hat{}) \leq m$. Thus $s(\mathcal{Q}(\sqrt{t}), \hat{}) = s(\mathcal{Q}, \hat{})$. If we take the homogeneous equation obtained from equation (3), the same reasoning as above, allows us to conclude that $s(\mathcal{O}, -) = s(\mathcal{Q}, -)$. ∎

Remark 4.2 Pumplün and Unger ([4], 3.6) show that for every octonion algebra $\mathcal{O}$, $s(\mathcal{O}, -)$ is a power of two and any prescribed power of two occurs as the level of an octonion division algebra with standard involution, see p. 516. We have

Corollary 4.0A There exist algebra $\mathcal{Q}(\sqrt{c})$ and octonion algebra $\mathcal{O}$ with $s(\mathcal{Q}(\sqrt{c}), \hat{})$ and $s(\mathcal{O}, -)$ being any prescribed power of two.

Proof: From ([2], Proposition 1.5, Comment 2 p.470, and Proposition 2.4) the results follows. ∎

Let F be a C_i-field (for the definition see [5], p. 97), $\mathcal{Q} = \big((a, b)/F\big)$ be a quaternion division algebra. By simple application of the Lang-Nagata Theorem ([5], p.100), ([2], Proposition 3.1) showed that the level of $(\mathcal{Q}, \hat{})$ is less than or equal to 3.2^{i-2}, $(i \geq 2)$. Similarly, the same theorem implies that the set of seven homogeneous equations in $7n$ variables obtained from (5) has a common zero if $8n > 7.2^i$. Hence $s(\mathcal{Q}(\sqrt{c}), \hat{}) \leq 7.2^{i-3}$. But, we see that, if there exists a solution for equation (1), we will take $\gamma_r = \delta_r = 0$ to get a solution for (4). Similarly, a solution of (4) can be completed to a solution of (5), by taking $x_{1r} = \alpha_r$, $x_{2r} = \beta_r$, $x_{3r} = \gamma_r$, $x_{4r} = \delta_r$ and $y_{ir} = 0$, for all i and r. Indeed, since $F(\sqrt{a}) \simeq F(i)$, and $F(i) \subset \mathcal{Q} \subset \mathcal{Q}(\sqrt{c})$, if $s\big(F(i), -\big)$ is finite, then $s(\mathcal{Q}(\sqrt{c}), \hat{}) \leq s(\mathcal{Q}, \hat{}) \leq s(F(\sqrt{a}), -)$. This implies the following

Proposition 4.2 Let F be a C_i-field, $\mathcal{Q} = \big((a, b)/F\big)$ a quaternion algebra and $c \in \dot{F}$. Then $s(\mathcal{Q}(\sqrt{c}), \hat{}) \leq s(\mathcal{Q}, \hat{}) \leq s\big(F(\sqrt{a}), -\big) \leq 2^{i-1}$.

Proof: It remains to show that $s\big(F(\sqrt{a}), -\big) \leq 2^{i-1}$. Since F is a C_i-field, the equation $\sum_{r=1}^{n}(\alpha_r^2 - a\beta_r^2) = 0$ in $2n$ variables has a non-trivial zero, if $2n > 2^i$. From Lemma (2.2) the results follows. ∎

It is easy to relate the hermitian level of $(\mathcal{Q}(\sqrt{c}), \hat{})$ to the u-invariant of the base field F. The u-invariant of F, $u(F)$ is the maximal dimension of an anisotropic quadratic form over F. See [1] and [5], for more on the u-invariant.

Proposition 4.3 Let $\mathcal{Q}$ be as in Proposition 4.2. Then $s(\mathcal{Q}(\sqrt{c}), \widehat{\ }) \leq 28u(F)$.

Proof: Recall that, for a natural number r, the number $u_r(F)$ is defined to be the maximal number m for which there exists an m-dimensional F-vectorspace with r quadratic forms $q_1,\ q_2, \ldots, q_r$ having no common zero $u \neq 0$. In particular $u_1(F) = u(F)$. Clearly $s(\mathcal{Q}(\sqrt{c}), \widehat{\ }) \leq u_7(F)$. By Leep's results ([5], Chapter 2, 16.3 and 16.4) $u_r(F) \leq \frac{1}{2}r(r+1)u(F)$ thus giving $s(\mathcal{Q}(\sqrt{c}), \widehat{\ }) \leq 28u(F)$, in our case. ∎

References

1. Lam, T. Y., *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics 67, American Mathematical Society, Providence, Rhode Island, (2005).

2. Lewis, D. W., *Sums of hermitian squares*, J. of Algebra 115, 466-480, (1988).

3. O'Shea, J., *Levels and sublevels of composition algebras*, Indag. Mathem. 18(1), 147-159, (2007).

4. Pumplün, S. and Unger, T., *The hermitian level of composition algebras*, Manuscripta Math. 109, 511-525, (2002).

5. Scharlau, W., *Quadratic and Hermitian Forms*, Springer-Verlag, Heidelberg / New York / Tokyo, (1985).

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