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Schwarz rearrangement does not decrease the energy for the pseudo p-Laplacian operator

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ABSTRACT: It is well known that the Schwarz symmetrization decrease the energy for the p-Laplacian operator, i.e

$$\int_{\Omega} |\nabla u|^p \, dx \geq \int_{\Omega^\star} |\nabla u^\star|^p \, dx.$$

where $u^\star$ is the Schwarz rearranged function of u, for appropriate u and Ω. In this note, we shall proof that the Schwarz rearrangement does not decrease the energy for the pseudo p-Laplacian operator, that is, there exist a bounded domain $\Omega \subset \mathbb{R}^N$ and a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega^\star} \sum_{i=1}^{n} \left| \frac{\partial u^\star}{\partial x_i} \right|^p dx \geq \int_{\Omega} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^p dx.$$

Key Words: Schwarz symmetrization, pseudo p-Laplacian operator.

Contents

1. Introduction

The rearrangement method is defined by replacing a given function u by a related function $u^\star$ wich has some properties like monotonicity or symmetry. The function $u^\star$ can be reconstructed from its level sets

$$\Omega_c = \{x \in \Omega \mid u(x) \geq c\}.$$

1.1. A CATALOGUE OF REARRANGEMENT. In the litterature we find many type of rearrangement,

1. circular and spherical symmetrization,
2. monotone decreasing rearrangement in direction y,
3. radial symmetrization,
4. Schwarz symmetrization,

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5. Steiner symmetrization in direction y.

more details of rearrangement can be found in [3,4,5,7]. In this note we are intersted only by *Schwarz symmetrization* (the most frequently used kind symmetrization). Hence, for a Lebesgue measurable set $D \subset \mathbb{R}^n$ we define the *Schwarz symmetrization* $D^\star$ of D by

$$D^\star = \begin{cases} B(0,R) & if \quad D \neq \emptyset \\ \emptyset & if \quad D = \emptyset \end{cases}$$

where $B(0,R)$ is a ball of $\mathbb{R}^n$ with center in the origin with same $n-dim$. Lebesgue measure and for a Lipschitz continuous function u, the rearranged function $u^\star$ is defined as follows

$$u^\star(x) = \sup\{c \in \mathbb{R} | \quad x \in \Omega_c^\star\} \text{ for } x \in \Omega^\star$$

1.2. Some results for Schwarz symmetrization. One of the first powerful applications of Schwarz symmetrization was the proof of the *Krahn-Faber* inequality [6] : Among all fixed membranes of given area, the circular one has the lowest principal eigenvalue. This was shown by looking at

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{|u|^2}.$$

one can easily conclude, with the two following propositions, that $\lambda_1(\Omega) \geq \lambda_1(\Omega^\star)$

Proposition 1.1 *([1,5,7]) For every continuous mapping $F : \mathbb{R}^+ \to \mathbb{R}$ and every nonnegative function $u : \overline{\Omega} \to \mathbb{R}^+$ then,*

$$\int_{\overline{\Omega}} F(u)\, dx = \int_{\overline{\Omega}^\star} F(u^\star)\, dx.$$

Proposition 1.2 *([1,5,7]) For $u \neq 0 \in W_0^{1,p}(\Omega)$ and $p > 1$, we have*

$$E_p(u) = \frac{1}{p}\int_{\overline{\Omega}} |\nabla u|^p\, dx \geq \frac{1}{p}\int_{\overline{\Omega}^\star} |\nabla u^\star|^p\, dx = E_p(u^\star)$$

Then, Schwarz symmetrization decrease the potentiel energy $E_p(u)$ of p-Laplacian operator $\Delta_p u$ defined by

$$\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right).$$

The same question is posed for the so called *pseudo p-Laplacian operator* defined by

$$\Delta_p^{'} u = div\left(\sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right).$$

We show, in this paper, that the answer is negative by exhibiting an explicit function with Schwarz's symmetrization does not decrease the energy for pseudo p-Laplacian operator.

2. Main result

Theorem 2.1 *The Schwarz rearrangement does not decrease the energy for the pseudo p-Laplacian operator, that is, there exist a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $u \in W_0^{1,p}(\Omega)$ such that*

$$\sum_{i=1}^{n} \int_{\Omega^\star} \left| \frac{\partial u^\star}{\partial x_i} \right|^p dx \geq \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p dx. \tag{1}$$

where $\Omega^\star$ and $u^\star$ are the Schwarz rearrangement of Ω and u respectively.

Proof. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | \quad 0 \leq |x_1| + |x_2| \leq \sqrt{\pi}\} \text{ and } u(x_1, x_2) = \sqrt{\pi} - (|x_1| + |x_2|),$$

u is a Lipschitz continuous function that $u \in W_0^{1,p}(\Omega)$. Then the Schwarz rearrangement of Ω is $\Omega^\star = B(0, \sqrt{2})$.
Level sets of u are given by

$$\Omega_c = \{(x_1, x_2) \in \mathbb{R}^2 | \quad |x_1| + |x_2| < \sqrt{\pi} - c\}.$$

Then, $|\Omega_c| = meas(\Omega_c) = 2(\sqrt{\pi} - c)^2$, so

$$u^\star(x_1, x_2) = \sqrt{\pi} - \sqrt{\frac{\pi}{2}} (x_1^2 + x_2^2)^{\frac{1}{2}}.$$

Now we have

$$\left| \frac{\partial u}{\partial x_1} \right| = 1 = \left| \frac{\partial u}{\partial x_2} \right|,$$

then,

$$\sum_{i=1}^{2} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p dx = 2meas(\Omega) = 4\pi \tag{2}$$

in the other hand,

$$\left| \frac{\partial u^\star}{\partial x_1} \right| = \sqrt{\frac{\pi}{2}} \frac{|x_1|}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \quad \text{and} \quad \left| \frac{\partial u^\star}{\partial x_2} \right| = \sqrt{\frac{\pi}{2}} \frac{|x_2|}{(x_1^2 + x_2^2)^{\frac{1}{2}}}$$

by passing to polar coordinates we obtain

$$\begin{aligned}
\sum_{i=1}^{2} \int_{\Omega^\star} \left| \frac{\partial u^\star}{\partial x_i} \right|^p dx &= \left(\frac{\pi}{2}\right)^{\frac{p}{2}} \left(\int_0^{2\pi} |\cos\theta|^p d\theta \int_0^{\sqrt{2}} r\, dr + \int_0^{2\pi} |\sin\theta|^p d\theta \int_0^{\sqrt{2}} r\, dr \right) \\
&= \left(\frac{\pi}{2}\right)^{\frac{p}{2}} \left(\int_0^{2\pi} |\cos\theta|^p\, d\theta + \int_0^{2\pi} |\sin\theta|^p d\theta \right) \left[\frac{1}{2} r^2\right]_0^{\sqrt{2}} \\
&= \left(\frac{\pi}{2}\right)^{\frac{p}{2}} \left(\int_0^{2\pi} |\cos\theta|^p\, d\theta + \int_0^{2\pi} |\sin\theta|^p d\theta \right) \\
&= 4\left(\frac{\pi}{2}\right)^{\frac{p}{2}} \left(\int_0^{\frac{\pi}{2}} |\cos\theta|^p\, d\theta + \int_0^{\frac{\pi}{2}} |\sin\theta|^p d\theta \right) \\
&= 8\left(\frac{\pi}{2}\right)^{\frac{p}{2}} \int_0^{\frac{\pi}{2}} |\sin\theta|^p d\theta \quad \left(\int_0^{\frac{\pi}{2}} |\cos\theta|^p\, d\theta = \int_0^{\frac{\pi}{2}} |\sin\theta|^p\, d\theta \right)
\end{aligned}$$

we have used the following equality

$$\begin{aligned}
\int_0^{2\pi} |\cos\theta|^p \, d\theta &= \int_0^{2\pi} |\sin\theta|^p \, d\theta \\
&= \int_0^{\pi} |\sin\theta|^p \, d\theta + \int_{\pi}^{2\pi} |\sin\theta|^p \, d\theta \\
&= 2\int_0^{\pi} |\sin\theta|^p \, d\theta \quad \text{(put in the second integral } \theta' = \theta - \pi) \\
&= 2\left(\int_0^{\frac{\pi}{2}} |\sin\theta|^p \, d\theta + \int_{\frac{\pi}{2}}^{\pi} |\sin\theta|^p \, d\theta\right) \\
&= 2\left(\int_0^{\frac{\pi}{2}} |\sin\theta|^p \, d\theta + \int_0^{\frac{\pi}{2}} |\sin\theta|^p \, d\theta\right) \quad \text{(put in the second integral } \theta' = \pi - \theta) \\
&= 4\int_0^{\frac{\pi}{2}} |\sin\theta|^p \, d\theta,
\end{aligned}$$

so,

$$\sum_{i=1}^{2} \int_{\Omega^\star} \left|\frac{\partial u^\star}{\partial x_i}\right|^p dx = 8\left(\frac{\pi}{2}\right)^{\frac{p}{2}} \int_0^{\frac{\pi}{2}} |\sin\theta|^p \, d\theta \tag{3}$$

the integral in the right member of equation (3) is given by the well known Wallis formula [2] (page : 15)

$$\int_0^{\frac{\pi}{2}} |\sin\theta|^p \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}.$$

Finally,

$$\sum_{i=1}^{2} \int_{\Omega^\star} \left|\frac{\partial u^\star}{\partial x_i}\right|^p dx = 4\left(\frac{\pi}{2}\right)^{\frac{p}{2}} \sqrt{\pi} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(\frac{p}{2}+1)} \tag{4}$$

the function $\Gamma(\alpha)$ is increasing for $\alpha \geq \frac{3}{2}$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for all $\alpha > -1$ so,

$$\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(\frac{p}{2}+1)} \geq \frac{\Gamma\left(\frac{p}{2}\right)}{\frac{p}{2}\Gamma(\frac{p}{2})} = \frac{2}{p},$$

applying equation (4) to get

$$\sum_{i=1}^{2} \int_{\Omega^\star} \left|\frac{\partial u^\star}{\partial x_i}\right|^p dx \geq \frac{8}{p}\left(\frac{\pi}{2}\right)^{\frac{p}{2}} \sqrt{\pi}$$

then the inequality

$$\sum_{i=1}^{2} \int_{\Omega^\star} \left|\frac{\partial u^\star}{\partial x_i}\right|^p dx \geq \sum_{i=1}^{2} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^p dx$$

is verified if

$$\frac{8}{p}\left(\frac{\pi}{2}\right)^{\frac{p}{2}} \sqrt{\pi} \geq 4\pi$$

which is equivalent to

$$\left(\frac{\pi}{2}\right)^{\frac{p}{2}} - \frac{p}{2}\sqrt{\pi} \geq 0. \tag{5}$$

An elementary study of the function ($x = \frac{p}{2}$)

$$f(x) = \left(\frac{\pi}{2}\right)^{x} - x\sqrt{\pi}$$

shows that f is strictly increasing in $[\frac{1}{\ln 2}, +\infty[$. Equation (5) is then established if $f(\frac{p}{2}) \geq 0$, that is $p \geq p_c$ where p_c is defined by $f(\frac{p_c}{2}) = 0$. Mean value theorem shows that $p_c \in [9, 10]$. Consequently, for all $p \geq 10$

$$\sum_{i=1}^{2} \int_{\Omega^\star} \left|\frac{\partial u^\star}{\partial x_i}\right|^p dx \geq \sum_{i=1}^{2} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^p dx.$$

Conclusion : the Schwarz rearrangement does not decrease the energy for the pseudo p-Laplacian operator like it does for p-Laplacian operator.

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