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A non resonance under and between the two first eigenvalues in a nonlinear boundary problem

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ABSTRACT: In this paper we study the non resonance of solutions under and between the two first eigenvalues for the problem

$$\begin{array}{rcll} \Delta_p u & = & |u|^{p-2}u & \text{in} \quad \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} & = & f(x,u) & \text{on} \quad \partial\Omega. \end{array}$$

Key Words:: p-Laplacian, Nonlinear boundary conditions, Non resonance.

Contents

1. Introduction

Consider the following nonlinear boundary problem

$$\begin{array}{rcll} \Delta_p u & = & |u|^{p-2}u & \text{in} \quad \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} & = & g(x,u) & \text{on} \quad \partial\Omega, \end{array}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $p>1$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\dfrac{\partial}{\partial \nu}$ is the outer normal derivative.

The case $g(x,u) = \lambda V(x)|u|^{p-2}u$, where V is the weight such that

$$V^+ \neq 0 \text{ on } \partial\Omega \quad \text{and} \quad V \in L^s(\partial\Omega), \tag{1.1}$$

where $s > \frac{N-1}{p-1}$ if $1<p\le N$ and $s \ge 1$ if $N<p$, has been treated by J.F.Bonder and J.D.Rossi in [3], they have proved that there exists a sequence of variational

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eigenvalues $\lambda_k \to +\infty$, defined by

$$(\lambda_k(V))^{-1} = \sup_{C \in C_k} \min_{u \in C} \frac{\int_{\partial\Omega} |u|^p V(x) \partial\sigma}{\|u\|^p_{W^{1,p}(\Omega)}}, \tag{1.2}$$

where $C_k = \{C \subset W^{1,p}(\Omega) ;\ C \text{ is compact, symmetric and } \gamma(C) \geq k\}$ and γ is the genus's function. The authors have also proved that $\lambda_1(V)$ is the first eigenvalue, isolated, simple and monotone with respect to the weight, and it's defined as $\lambda_1(V) = \min\left\{ \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\int_{\partial\Omega}|u|^p V(x)\partial\sigma} : u \in W^{1,p}(\Omega) \right\}$, $\lambda_2(V)$ is the seconde one characterized by

$$(\lambda_2(V))^{-1} = \sup\left\{ \int_{\partial\Omega} |u|^p V(x) \partial\sigma : \|u\|^p_{W^{1,p}(\Omega)} = 1 \quad \text{and} \quad u \in A \right\}, \tag{1.3}$$

where

$$A = \{u \in W^{1,p}(\Omega) : |\partial\Omega^{\pm}(u)| \geq c(V)\} \quad \text{if} \quad s > 1 \text{ or } 1 < p \leq N,$$

and

$$A = \left\{ u \in W^{1,p}(\Omega) : \int_{\partial\Omega^{\pm}} |V(x)| \partial\sigma \geq d(V) \right\} \quad \text{if} \quad p > N \text{ and } s = 1,$$

with $\partial\Omega^+(u) = \partial\Omega \cap \{u > 0\}$, $\partial\Omega^-(u) = \partial\Omega \cap \{u < 0\}$, $|B| = meas_\sigma(B)$ denotes the $N-1$ dimensional measure of a subset $B \subset \partial\Omega$, $c(V) = \left(\frac{S_{p^*}}{\lambda_2(V)\|V\|_{L^s(\partial\Omega)}}\right)^\eta$, $d(V) = \frac{S_\infty}{\lambda_2(V)}$ where S_q is the best constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, $p^* = \frac{p(N-1)}{N-p}$ for $1 < p < N$, $p^* = \infty$ for $p \geq N$, $\eta = \frac{s(N-1)}{sp-N}$ for $1 < p \leq N$ and $\eta = 2s'$ for $p > N$ and $s > 1$, here s' is the conjugate of s. This problem will be named $P(V)$.

In [2], one has proved that, in the case $g(x,u) = \lambda V(x)|u|^{p-2}u + h$ with V satisfies the same last conditions and $h \in L^s(\partial\Omega)$, the solutions are in $C^{1,\alpha}(\overline{\Omega})$ for some α in $]0,1[$. Now we will study the case $g(x,u) = f(x,u) + h$, with $h \in L^{p'}(\partial\Omega)$ where p' is the conjugate of p and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, we show a non resonance of solutions under and between the two first eigenvalues.

2. Main results

In the theorems that follow we study a monotonicity of the two first eigenvalues with respect to the weight. One consider two weight's functions V_1 and V_2 satisfying the condition (1.1). Without loss of generality, one can assume that the weights are in the same space $L^s(\partial\Omega)$.

Theorem 2.1 *If* $V_1(x) \lneqq V_2(x)$ *a.e in* $\partial\Omega$ *then* $\lambda_1(V_1) > \lambda_1(V_2)$.

Theorem 2.2 *If* $V_1(x) \leq V_2(x)$ *a.e in* $\partial\Omega$ *and if one of this conditions is satisfied*

(i) *$s > 1$ or $1 < p \leq N$ with* $|\partial\Omega \cap \{V_1 = V_2\}| < c(V_1)$,

(ii) *$s = 1$ and $p > N$ with* $\int_{\partial\Omega \cap \{V_1 = V_2\}} |V_1(x)|\,\partial\sigma < d(V_1)$,

then $\lambda_2(V_1) > \lambda_2(V_2)$.

Remark 2.1 *The notation $\lneqq$ means that one has a large inequality a.e in $\partial\Omega$ and a strict inequality in a subset with a positive measure.*

In the theorems 2.4 and 2.5 we prove the existence of solutions to the problem

$$\begin{array}{rcll} \Delta_p u & = & |u|^{p-2}\,u & \text{in} \quad \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} & = & f(x,u) + h & \text{on} \quad \partial\Omega, \end{array} \tag{2.1}$$

where $h \in L^{p'}(\partial\Omega)$, p' is the conjugate of p and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, with conditions on the behavior of the ratios $\frac{f(x,s)}{s|s|^{p-2}}$ and $p\frac{F(x,s)}{|s|^p}$ under the first eigenvalue and between the two first eigenvalues of the problem

$$\begin{array}{rcll} \Delta_p u & = & |u|^{p-2}\,u & \text{in} \quad \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} & = & \lambda\,|u|^{p-2}\,u & \text{on} \quad \partial\Omega. \end{array} \tag{2.2}$$

Consider the following conditions

(h1) $\forall R > 0, \exists \Phi_R \in L^{p'}(\partial\Omega)$ such that $\max\limits_{|s|\leq R} |f(x,s)| \leq \Phi_R(x)$ a.e in $\partial\Omega$.

(h2) $\lambda_1 \leq l(x) := \liminf\limits_{|s|\to+\infty} \frac{f(x,s)}{s|s|^{p-2}} \leq k(x) := \limsup\limits_{|s|\to+\infty} \frac{f(x,s)}{s|s|^{p-2}} \leq \lambda_2$ a.e in $\partial\Omega$.

(h3) $\lambda_1 \lneqq L(x) := p\liminf\limits_{|s|\to+\infty} \frac{F(x,s)}{|s|^p} \leq K(x) := p\limsup\limits_{|s|\to+\infty} \frac{F(x,s)}{|s|^p} \leq \lambda_2$ a.e in $\partial\Omega$, with

$$meas_\sigma\{x \in \partial\Omega : K(x) = \lambda_2\} < 2\left(\frac{S_{P^*}}{\lambda_2}\right)^\eta, \tag{2.3}$$

where $\eta = \frac{N}{p}$ if $1 < p \leq N$ and $\eta = 2$ if $p > N$, and $F : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $F(x,s) = \int_0^s f(x,t)\,dt$.

(h2') $-\infty < l(x) := \liminf\limits_{|s|\to+\infty} \frac{f(x,s)}{s|s|^{p-2}} \leq k(x) := \limsup\limits_{|s|\to+\infty} \frac{f(x,s)}{s|s|^{p-2}} \leq \lambda_1$ a.e in $\partial\Omega$.

(h3') $K(x) := p\limsup\limits_{|s|\to+\infty} \frac{F(x,s)}{|s|^p} \lneqq \lambda_1$ and $K^+ \gneqq 0$ a.e in $\partial\Omega$.

One shows the following results

Theorem 2.3 *If V is a weight in $L^{\infty}(\partial\Omega)$ with $\lambda_1 \lneqq V(x) \leq \lambda_2$ a.e in $\partial\Omega$, and if the problem $P(V)$ admits a non trivial solution u, then*

$$meas_{\sigma}\{x \in \partial\Omega : V(x) = \lambda_2\} \geq 2\left(\frac{S_{P^*}}{\|V\|_{L^{\infty}(\partial\Omega)}}\right)^{\eta},$$

and u is an eigenfunction associated to λ_2.

Theorem 2.4 *Under the conditions (h1), (h2) and (h3), the problem (2.1) admits at least one solution characterized by a min-max principle.*

Theorem 2.5 *If (h1), (h2') and (h3') are satisfied then the problem (2.1) admits at least one solution.*

Remark 2.2 *1) The hypothesis (h2), (h3), (h2') and (h3') mean that $\forall\varepsilon > 0$ there exists $b_{\varepsilon} \in L^{p'}(\partial\Omega)$ and $d_{\varepsilon} \in L^{1}(\partial\Omega)$ such that a.e in $\partial\Omega$ and $\forall s \in \mathbb{R}$ one has*

$$-b_{\varepsilon}(x) + (l(x) - \varepsilon)|s|^{p} \leq sf(x,s) \leq (k(x) + \varepsilon)|s|^{p} + b_{\varepsilon}(x), \tag{2.4}$$

and

$$-d_{\varepsilon}(x) + (L(x) - \varepsilon)\frac{|s|^{p}}{p} \leq F(x,s) \leq (K(x) + \varepsilon)\frac{|s|^{p}}{p} + d_{\varepsilon}(x), \tag{2.5}$$

2) If (h1) and (h2) or (h1) and (h2') are satisfied then there exists a real $a > 0$ and a function $b \in L^{p'}(\partial\Omega)$ such that a.e in $\partial\Omega$ and $\forall s \in \mathbb{R}$ one has

$$|f(x,s)| \leq a|s|^{p-1} + b(x). \tag{2.6}$$

We have to use the theorems

Theorem 2.6 *(see [1]) Let $\Phi \in C^{1}(X,\mathbb{R})$ be a functional satisfying the palais-smale condition (PS) in a Banach space X, $Q_0 \subset X \setminus \{0\}$ a symmetric compact and $E \subset X$ a nonempty symmetric set . If the following conditions are satisfied*

(P1) *If $Q \subset X \setminus \{0\}$ is a symmetric compact and $\gamma(Q) \geq \theta(Q_0)+1$, then $Q \cap E \neq \emptyset$.*

(P2) $\alpha := \max\limits_{Q_0}\Phi < \inf\limits_{E}\Phi := \beta$,

then $c := \inf\limits_{h\in\Gamma}\max\limits_{u\in h(\overline{D})}\Phi$ is a critical value to the functional Φ, where $D = co(Q_0)$ is the convex envelope of Q_0 and $\Gamma = \left\{h \in C\left(\overline{D}, X \setminus \{0\}\right) : h = id \text{ on } Q_0\right\}$. Moreover $c \geq \beta$.

Theorem 2.7 *Let X be a Banach space reflexive and $\Phi : X \longrightarrow \mathbb{R}$ a functional satisfying*

(i) *Φ is weakly lower semi-continuous,*

(ii) *Φ is coercive,*

then Φ attends his minimum.

Remark 2.3 *In theorem 2.6, $\theta(F)$ is defined for a closed and symmetric subset F in $X \setminus \{0\}$ by:*

$$\theta(F) \quad := \quad \sup \; \left\{k \in \mathbb{N} \;/\; \exists\, f: \; S^{k-1} \quad \longrightarrow \quad F \quad \textit{continuous and odd}\right\}$$

where $S^{k-1} = \left\{x \in \mathbb{R}^k : \|x\|_{\mathbb{R}^k} = 1\right\}$ if $F \neq \emptyset$, and $\theta(\emptyset) \;=\; 0$.

3. Proofs of theorems

3.1. PROOF OF THEOREM 2.1. Let u_1 be an eigenfunction associated to $\lambda_1(V)$ then

$$\lambda_1(V) \quad = \quad \frac{\|u_1\|^p_{W^{1,p}(\Omega)}}{\int_{\partial\Omega} |u_1|^p V(x)\,\partial\sigma},$$

and u_1 do not change sign in $\partial\Omega$. Supposing that $u_1 \geq 0$ in $\partial\Omega$, one show that $u_1 > 0$ on $\partial\Omega$. Indeed, if there exists $x \in \partial\Omega$ such that $u_1(x) = 0$, by the regularity proven in [2], $u_1 \in C^{1,\alpha}\left(\overline{\Omega}\right)$ and by the maximum principle of Vazquez

$$\frac{\partial u_1}{\partial \nu}(x) < 0,$$

so

$$0 > |\nabla u_1|^{p-2} \frac{\partial u_1}{\partial \nu}(x) = \lambda_1(V)\, V(x)\, |u_1(x)|^{p-2} u_1(x) = 0,$$

witch is impossible. Let V_1 and V_2 be two weight's functions such that for a.e in $\partial\Omega$ one has $V_1(x) \lneqq V_2(x)$ and u_1 be an eigenfunction associated to $\lambda_1(V_2)$, then

$$(\lambda_1(V_1))^{-1} \quad = \quad \frac{\int_{\partial\Omega} |u_1|^p V_1(x)\,\partial\sigma}{\|u_1\|^p_{W^{1,p}(\Omega)}}.$$

Since $u_1(x) \neq 0$ for all $x \in \partial\Omega$ and $V_1(x) \lneqq V_2(x)$ a.e in $\partial\Omega$ one has

$$\frac{\int_{\partial\Omega} |u_1|^p V_1(x)\,\partial\sigma}{\|u_1\|^p_{W^{1,p}(\Omega)}} \quad < \quad \frac{\int_{\partial\Omega} |u_1|^p V_2(x)\,\partial\sigma}{\|u_1\|^p_{W^{1,p}(\Omega)}},$$

also

$$\frac{\int_{\partial\Omega} |u_1|^p V_2(x)\,\partial\sigma}{\|u_1\|^p_{W^{1,p}(\Omega)}} \leq \sup \left\{ \frac{\int_{\partial\Omega} |u|^p V_2(x)\,\partial\sigma}{\|u\|^p_{W^{1,p}(\Omega)}} : u \in W^{1,p}(\Omega) \right\} = (\lambda_1(V_2))^{-1},$$

consequently

$$(\lambda_1(V_1))^{-1} \quad < \quad (\lambda_1(V_2))^{-1}.$$

3.2. PROOF OF THEOREM 2.2. One takes V_1 and V_2 such that $V_1(x) \leq V_2(x)$ a.e in $\partial\Omega$ and u_2 an eigenfunction associated to $\lambda_2(V_1)$,then for all $v \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla v dx + \int_\Omega |u_2|^{p-2} u_2 v dx = \lambda_2(V_1) \int_{\partial\Omega} |u_2|^{p-2} u_2 V_1(x) v \partial\sigma. \tag{3.1}$$

One considers $U_2^{\pm} = \overline{span\left\{u_2^+, u_2^-\right\}}$, $C = U_2^{\pm} \cap \{u \in W^{1,p}(\Omega) : \|u\|^p_{W^{1,p}(\Omega)} = p\}$, and $v = \alpha u_2^+ + \beta u_2^- \in C$. Applying the equality (3.1) at u_2^+ and u_2^- one finds

$$\int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla u_2^+ dx + \int_\Omega |u_2|^{p-2} u_2 u_2^+ dx = \lambda_2(V_1) \int_{\partial\Omega} |u_2|^{p-2} u_2 u_2^+ V_1(x) \partial\sigma$$

and

$$\int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla u_2^- dx + \int_\Omega |u_2|^{p-2} u_2 u_2^- dx = \lambda_2(V_1) \int_{\partial\Omega} |u_2|^{p-2} u_2 u_2^- V_1(x) \partial\sigma,$$

that means that

$$\int_\Omega \left|\nabla u_2^+\right|^p dx + \int_\Omega \left|u_2^+\right|^p dx = \lambda_2(V_1) \int_{\partial\Omega} \left|u_2^+\right|^p V_1(x) \partial\sigma,$$

and

$$\int_\Omega \left|\nabla u_2^-\right|^p dx + \int_\Omega \left|u_2^-\right|^p dx = \lambda_2(V_1) \int_{\partial\Omega} \left|u_2^-\right|^p V_1(x) \partial\sigma.$$

It's clear that $\gamma(C) = 2$, and $|v|^p = |\alpha|^p \left|u_2^+\right|^p + |\beta|^p \left|u_2^-\right|^p$, so

$$\int_\Omega |\nabla v|^p dx + \int_\Omega |v|^p dx = \lambda_2(V_1) \int_{\partial\Omega} |v|^p V_1(x) \partial\sigma.$$

But $\|v\|^p_{W^{1,p}(\Omega)} = p$, then $\frac{1}{\lambda_2(V_1)} = \frac{1}{p} \int_{\partial\Omega} |v|^p V_1 \partial\sigma$, this's true for all $v \in C$. Let

$$\zeta = \min \left\{ \frac{\int_{\partial\Omega} |u|^p V_2(x) \partial\sigma}{\|u\|^p_{W^{1,p}(\Omega)}} \quad / \quad u \in C \right\}.$$

One poses $a = \int_{\partial\Omega} \left|u_2^+\right|^p V_2(x) \partial\sigma$, $b = \int_{\partial\Omega} \left|u_2^-\right|^p V_2(x) \partial\sigma$, $c = \left\|u_2^+\right\|^p_{W^{1,p}(\Omega)}$ and $d = \left\|u_2^-\right\|^p_{W^{1,p}(\Omega)}$, then one has

$$\begin{aligned} \zeta &= \min_{(\alpha,\beta)\in\mathbb{R}^2} \left\{ \tfrac{1}{p} \int_{\partial\Omega} |u|^p V_2(x) \partial\sigma : u = \alpha u_2^+ + \beta u_2^- \text{ and } \|u\|^p_{W^{1,p}(\Omega)} = p \right\} \\ &= \min_{(\alpha,\beta)\in\mathbb{R}^2} \left\{ \tfrac{1}{p} \left(a|\alpha|^p + b|\beta|^p\right) : c|\alpha|^p + d|\beta|^p = p \right\}. \end{aligned}$$

By the Lagrange's theorem about the extremum, this one is attained for $\alpha = 0$ or $\beta = 0$, i.e

$$\zeta = \min\left(\frac{a}{c}, \frac{b}{d}\right),$$

and it's attained for $u_0 = \pm\sqrt[p]{p}\frac{u_2^+}{\|u_2^+\|_{W^{1,p}(\Omega)}}$ or $u_0 = \pm\sqrt[p]{p}\frac{u_2^-}{\|u_2^-\|_{W^{1,p}(\Omega)}}$. Thus we obtain

$$\frac{1}{\lambda_2(V_1)} = \frac{1}{p}\int_{\partial\Omega} |u_0|^p V_1(x)\,\partial\sigma$$

and

$$\frac{1}{p}\int_{\partial\Omega} |u_0|^p V_2(x)\,\partial\sigma = \min_{u\in C}\frac{\int_{\partial\Omega} |u|^p V_2(x)\,\partial\sigma}{\|u\|^p_{W^{1,p}(\Omega)}},$$

with $\{x \in \partial\Omega : u_0(x) \neq 0\}$ is either $\partial\Omega^+(u_2)$ or $\partial\Omega^-(u_2)$.

- Under the condition (i).

With the hypothesis $|\{x \in \partial\Omega : V_1(x) = V_2(x)\}| < c(V_1)$ and since $|\partial\Omega^\pm(u_2)| \geq c(V_1)$, one gets

$$|\{x \in \partial\Omega : V_1(x) < V_2(x)\} \cap \{u_0(x) \neq 0\}| > 0.$$

Indeed, if not we have

$$c(V_1) > |\{x \in \partial\Omega : V_1(x) = V_2(x)\} \cap \{u_0 \neq 0\}| = |\{u_0 \neq 0\}| = \left|\partial\Omega^\pm(u_2)\right|,$$

that's not true.

- Under the condition (ii).

If $|\{x \in \partial\Omega : V_1(x) < V_2(x)\} \cap \{u_0(x) \neq 0\}| = 0$, then

$$d(V_1) > \int_{\partial\Omega\cap\{V_1=V_2\}} |V_1(x)|\,\partial\sigma \geq \int_{\partial\Omega\cap\{V_1=V_2\}\cap\{u_0\neq 0\}} |V_1(x)|\,\partial\sigma,$$

but

$$\int_{\partial\Omega\cap\{V_1=V_2\}\cap\{u_0\neq 0\}} |V_1(x)|\,\partial\sigma = \int_{\{u_0\neq 0\}} |V_1(x)|\,\partial\sigma = \int_{\partial\Omega^\pm(u_2)} |V_1(x)|\,\partial\sigma,$$

this contradicts the result $\int_{\partial\Omega^\pm(u_2)} |V_1(x)|\,\partial\sigma \geq d(V_1)$.

Then, if (i) or (ii), we obtain the following inequality

$$\frac{1}{p}\int_{\partial\Omega} |u_0|^p V_1(x)\,\partial\sigma < \frac{1}{p}\int_{\partial\Omega} |u_0|^p V_2(x)\,\partial\sigma$$

with $\gamma(C) = 2$, consequently

$$\min_{u\in C}\int_{\partial\Omega} |u|^p V_2(x)\,\partial\sigma \leq \sup_{C\in\Gamma_2}\min_{u\in C}\frac{\int_{\partial\Omega} |u|^p V_2(x)\,\partial\sigma}{\|u\|^p_{W^{1,p}(\Omega)}} = \frac{1}{\lambda_2(V_2)}.$$

Finally $\frac{1}{\lambda_2(V_1)} < \frac{1}{\lambda_2(V_2)}$.

3.3. PROOF OF THEOREM 2.3. V is taken such that $\lambda_1 \lneqq V(x)$ a.e in $\partial\Omega$, so, by the theorem 2.1, one has $\lambda_1(V) < \lambda_1(\lambda_1) = 1$, and if u is a non trivial solution to the problem $P(V)$, then 1 is an eigenvalue, thus u changes sign on $\partial\Omega$, and $|\partial\Omega^{\pm}(u)| \geq c(V) = \left(\frac{S_{P^*}}{\|V\|_{L^\infty(\partial\Omega)}}\right)^{\eta}$ and $|\partial\Omega(u) \cap \{u = 0\}| \geq c(V)$ where $\eta = \frac{N}{p}$ if $1 < p \leq N$ and $\eta = 2$ if $p > N$. From (1.3) we have

$$(\lambda_2)^{-1} = \sup\left\{\int_{\partial\Omega} |v|^p \, \partial\sigma : \|v\|^p_{W^{1,p}(\Omega)} = 1 \text{ and } \left|\partial\Omega^{\pm}(v)\right| \geq c(1) = \left(\frac{S_{P^*}}{\lambda_2}\right)^{\eta}\right\},$$

and $\|V\|_{L^\infty(\partial\Omega)} \leq \lambda_2$ implies that $c(V) \geq c(1)$, then

$$\frac{u}{\|u\|_{W^{1,p}(\Omega)}} \in \left\{v \in W^{1,p}(\Omega) : \|v\|^p_{W^{1,p}(\Omega)} = 1 \quad \text{and} \quad \left|\partial\Omega^{\pm}(v)\right| \geq c(1)\right\},$$

so

$$\lambda_2 \int_{\partial\Omega} |u|^p \, \partial\sigma \quad \leq \quad \|u\|^p_{W^{1,p}(\Omega)} \quad = \quad \int_{\partial\Omega} V(x) \, |u|^p \, \partial\sigma.$$

One deduct that $\int_{\partial\Omega} (\lambda_2 - V(x)) \, |u|^p \, \partial\sigma \leq 0$, but $V(x) \leq \lambda_2$ a.e in $\partial\Omega$, thus one has $V(x) = \lambda_2$ a.e in $\{x \in \partial\Omega : u(x) \neq 0\}$, consequently

$$meas_\sigma \{x \in \partial\Omega : V(x) = \lambda_2\} \geq 2\left(\frac{S_{P^*}}{\|V\|_{L^\infty(\partial\Omega)}}\right)^{\eta}.$$

In addition to this, we have for all w in $W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla w dx + \int_\Omega |u|^{p-2} uw = \int_{\partial\Omega} V(x) \, |u|^{p-2} uw = \lambda_2 \int_{\partial\Omega} |u|^{p-2} uw.$$

i.e u is an eigenfunction associated to λ_2.

3.4. PROOF OF THEOREM 2.4. One introduces the energy's function Φ associated to the problem (2.1)

$$\Phi(u) = \frac{1}{p} \|u\|^p_{W^{1,p}(\Omega)} - \int_{\partial\Omega} F(x, u) - \int_{\partial\Omega} hu \, \partial\sigma.$$

Under the conditions (h1), (h2) and (h3) Φ is well defined, C^1 and for all u and v in $W^{1,p}(\Omega)$

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_\Omega |u|^{p-2} uv\partial\sigma - \int_{\partial\Omega} f(x, u) \, v\partial\sigma - \int_{\partial\Omega} hv\partial\sigma. \tag{3.2}$$

i) Let us show that Φ is (PS)

By contradiction, we suppose that there exists a sequence $(u_n)_n$ in $W^{1,p}(\Omega)$ such that $\|u_n\|_{W^{1,p}(\Omega)} \to +\infty$, $\Phi(u_n)$ bounded and $\Phi'(u_n) \to 0$. One poses

$v_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$ then $\|v_n\|_{W^{1,p}(\Omega)} = 1$, so $(v_n)_n$ admits a subsequence, noted also, $(v_n)_n$ such that

$$\begin{array}{cccc} v_n & \rightharpoonup & v & \text{in} & W^{1,p}(\Omega) \\ v_n & \to & v & \text{in} & L^p(\Omega) \\ v_n & \to & v & \text{in} & L^p(\partial\Omega) \\ v_n(x) & \to & v(x) & \text{a.e.in} & \overline{\Omega} \end{array}$$

Applying the equality (3.2) at u_n and dividing by $\|u_n\|_{W^{1,p}(\Omega)}^{p-1}$, we obtain for all w in $W^{1,p}(\Omega)$

$$\begin{array}{rcl} \frac{\langle \Phi_p'(u_n), w\rangle}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} & = & \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla w dx + \int_\Omega |v_n|^{p-2} v_n w \partial\sigma \\ & - & \int_{\partial\Omega} \frac{f(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} w \partial\sigma - \int_{\partial\Omega} \frac{h}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} w \ \partial\sigma. \end{array} \tag{3.3}$$

Tending $n \to +\infty$ we remark that

$$\begin{array}{ccc} \frac{\langle \Phi_p'(u_n), w\rangle}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} & \to & 0 \\ \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla w + \int_\Omega |v_n|^{p-2} v_n w & \to & \int_\Omega |\nabla v|^{p-2} \nabla v \nabla w + \int_\Omega |v|^{p-2} v w \\ \int_{\partial\Omega} \frac{h}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} w \ \partial\sigma & \to & 0 \end{array}$$

and for $\int_{\partial\Omega} \frac{f(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} w \partial\sigma$, one has from (2.6) the sequence $\left(\frac{f(.,u_n(.))}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}}\right)_n$ is bounded in $L^{p'}(\partial\Omega)$, then for a subsequence $\left(\frac{f(.,u_n(.))}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}}\right)_n$ converges weakly in $L^{p'}(\partial\Omega)$ to a function $g \in L^{p'}(\partial\Omega)$.

Lemma 3.4.1 *If (2.6) then $g(x) = 0$ a.e in $\partial\Omega \cap \{v = 0\}$.*

Proof: Let $B = \partial\Omega \cap \{v = 0\}$, and consider the function test $T(x) = sign(g(x)) I_B$ where I_B is the characteristic function of B and $sign : \mathbb{R} \longrightarrow \{-1, 1\}$ such that $sign(x) = 1$ if $x \geq 0$ and $sign(x) = -1$ if $x < 0$. From (2.6) for $s = u_n(x)$, multiplying by $T(x)$ and dividing by $\|u_n\|_{W^{1,p}(\Omega)}^{p-1}$, one has a.e in $\partial\Omega$:

$$\left| \frac{f(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} T(x) \right| \leq a |v_n(x)|^{p-1} + \frac{b(x)}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} \to 0 \text{ in } L^{p'}(\partial\Omega),$$

then

$$\int_{\partial\Omega} \frac{f(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}^{p-1}} T(x) \partial\sigma \to \int_{\partial\Omega} g(x) T(x) \partial\sigma = \int_B |g(x)| \partial\sigma = 0,$$

so $g(x) = 0$ a.e in B. □

Let, now, consider the function $m(x)$ defined as

$$m(x) = \begin{cases} \frac{g(x)}{|v|^{p-2}v} & \text{in} \quad \partial\Omega \backslash B \\ \frac{1}{2}(\lambda_1 + \lambda_2) & \text{in} \quad B. \end{cases}$$

Lemma 3.4.2 *The operator $T : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ such that for all u and v in $W^{1,p}(\Omega)$: $\langle T(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx$, is monotone of type (S_+).*

Proof: For u and v in $W^{1,p}(\Omega)$ one has

$$\langle T(u), v\rangle \leq \|\nabla u\|_{L^p(\Omega)}^{p-1} \|\nabla v\|_{L^p(\Omega)},$$

and

$$\begin{aligned} \langle T(u) - T(v), u - v\rangle &= \|\nabla u\|_p^p + \|\nabla v\|_p^p - \langle T(u), v\rangle - \langle T(v), u\rangle \\ &\geq \left(\|\nabla u\|_p^{p-1} - \|\nabla v\|_p^{p-1}\right)\left(\|\nabla u\|_p - \|\nabla v\|_p\right), \end{aligned} \tag{3.4}$$

with $\|u\|_p = \|u\|_{L^p(\Omega)}$, so T is monotone. Let $(u_n)_n$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\limsup\limits_{n\to+\infty} \langle T(u_n), u_n - u\rangle \leq 0$, we will show that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. Remarking that $u_n \to u$ strongly in $L^p(\Omega)$, one conclude that $\|u_n - u\|_{L^p(\Omega)} \to 0$. From (3.4) one has

$$\langle T(u_n), u_n - u\rangle \geq \left(\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1}\right)\left(\|\nabla u_n\|_p - \|\nabla u\|_p\right) + \langle T(u), u_n - u\rangle,$$

since $u_n \rightharpoonup u$ then $\langle T(u), u_n - u\rangle \to 0$, and

$$\begin{aligned} 0 &\geq \limsup_{n\to+\infty} \langle T(u_n), u_n - u\rangle \\ &\geq \lim_{n\to+\infty} \left(\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1}\right)\left(\|\nabla u_n\|_p - \|\nabla u\|_p\right) \\ &\geq 0. \end{aligned}$$

Thus $\|\nabla u_n\|_{L^p(\Omega)} \to \|\nabla u\|_{L^p(\Omega)}$ and $\|u_n\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega)}$, then $\|u_n\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)}$. According to the propriety that $W^{1,p}(\Omega)$ is uniformly convexe, one conclude that $\|u_n - u\|_{W^{1,p}(\Omega)} \to 0$. □

Lemma 3.4.3 *The sequence $(v_n)_n$ is strongly convergent to v in $W^{1,p}(\Omega)$ witch's a solution of the problem $P(m)$*

$$\begin{aligned} \Delta_p v &= |v|^{p-2} v \quad \text{in} \quad \Omega \\ |\nabla v|^{p-2} \frac{\partial v}{\partial \nu} &= m(x) |v|^{p-2} v \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Proof: From (3.2), one has for all u and v in $W^{1,p}(\Omega)$

$$\left\langle \Phi_p'(u), v\right\rangle = \langle T(u), v\rangle + \int_\Omega |u|^{p-2} uv\partial\sigma - \int_{\partial\Omega} f(x,u)\, v\partial\sigma - \int_{\partial\Omega} hv\partial\sigma. \quad (3.5)$$

Replacing u by u_n, v by $v_n - v$ in (3.5) and dividing by $\|u_n\|^{p-1}_{W^{1,p}(\Omega)}$, one concluds

$$\langle T(v_n), v_n - v\rangle = \frac{\left\langle \Phi_p'(u_n), v_n - v\right\rangle}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} + \int_{\partial\Omega} \frac{f(x,u_n)(v_n - v)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} + \int_{\partial\Omega} \frac{h(v_n - v)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}}, \quad (3.6)$$

but $\Phi_p'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$, $\|u_n\|_{W^{1,p}(\Omega)} \to 0$, $v_n \to v$ in $L^p(\Omega)$ and $\frac{f(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \rightharpoonup m(x)|v|^{p-2} v$ weakly in $L^{p'}(\Omega)$ then we have

$$\lim_{n\to+\infty} \langle T(v_n), v_n - v\rangle = 0,$$

and since T is (S_+) one has $v_n \to v$ strongly in $W^{1,p}(\Omega)$, and

$$\|v_n\|_{W^{1,p}(\Omega)} = \|v\|_{W^{1,p}(\Omega)} = 1.$$

Moreover, tending n to $+\infty$ in (3.3) one obtains for all $w \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla w dx + \int_\Omega |v|^{p-2} vw dx = \int_{\partial\Omega} m(x)|v|^{p-2} vw\partial\sigma,$$

thus v is a solution to the problem $P(m)$, and $\|v\|_{W^{1,p}(\Omega)} = 1 = \int_{\partial\Omega} m(x)|v|^p \partial\sigma$. □

Lemma 3.4.4 *With (h2) one has $\lambda_1 \le m(x) \le \lambda_2$ a.e in $\partial\Omega$.*

Proof: It's easy to see that $\lambda_1 \le m(x) \le \lambda_2$ a.e in B, it remains to show that $\lambda_1 \le m(x) \le \lambda_2$ a.e in $\partial\Omega\backslash B$. For this, we consider the following subsets

$$D_1 = \{x \in \partial\Omega\backslash B : m(x) < \lambda_1\}, \quad D_2 = \{x \in \partial\Omega\backslash B : \lambda_2 < m(x)\}$$

and we prove that $meas_\sigma(D_1) = meas_\sigma(D_2) = 0$. Indeed, from (h2) one has

$$u_n(x) f(x, u_n(x)) \ge (\lambda_1 - \varepsilon)|u_n(x)|^p - b_\varepsilon(x),$$

one divides by $\|u_n\|^p_{W^{1,p}(\Omega)}$, and one integer on D_1, it comes

$$\int_{D_1} v_n(x) \frac{f(x,u_n(x))}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \partial\sigma \ge (\lambda_1 - \varepsilon)\int_{D_1} |v_n(x)|^p \partial\sigma - \int_{D_1} \frac{b_\varepsilon(x)}{\|u_n\|^p_{W^{1,p}(\Omega)}} \partial\sigma,$$

and when n tends to $+\infty$ one gets

$$\int_{D_1} v(x) g(x)\partial\sigma \quad \ge \quad (\lambda_1 - \varepsilon)\int_{D_1} |v(x)|^p \partial\sigma.$$

Since ε is arbitrary, one concludes that

$$0 \leq \int_{D_1} \left(v\left(x\right) g\left(x\right) - \lambda_1 \left|v\left(x\right)\right|^p\right) \partial\sigma = \int_{D_1} \left(m\left(x\right) - \lambda_1\right) \left|v\left(x\right)\right|^p \partial\sigma,$$

according to the definition of D_1, this inequality implies that $meas_\sigma\left(D_1\right) = 0$. By the same way one proves that $meas_\sigma\left(D_2\right) = 0$. □

Lemma 3.4.5 *From (h3) one has $\lambda_1 \lneqq m\left(x\right) < \lambda_2$ a.e in $\partial\Omega$.*

Proof: Let us show that $\lambda_1 \lneqq m\left(x\right)$ a.e in $\partial\Omega$. By contradiction we suppose that $m\left(x\right) = \lambda_1$ a.e $x \in \partial\Omega$, then v is an eigenfunction associated to λ_1, and one has by (h3), for all $\varepsilon > 0$, $\exists d_\varepsilon \in L^1\left(\partial\Omega\right)$ such that a.e in $\partial\Omega$

$$F\left(x, u_n\left(x\right)\right) \quad \geq \quad \left(L\left(x\right) - \varepsilon\right) \tfrac{\left|u_n(x)\right|^p}{p} - d_\varepsilon\left(x\right).$$

Then, one divides by $\|u_n\|^p_{W^{1,p}(\Omega)}$ and one integer, it comes

$$\int_{\partial\Omega} \frac{F\left(x, u_n\left(x\right)\right)}{\|u_n\|^p_{W^{1,p}(\Omega)}} \partial\sigma \geq \int_{\partial\Omega} \left(L\left(x\right) - \varepsilon\right) \frac{\left|v_n\left(x\right)\right|^p}{p} \partial\sigma - \int_{\partial\Omega} \frac{d_\varepsilon\left(x\right)}{\|u_n\|^p_{W^{1,p}(\Omega)}} \partial\sigma,$$

also

$$\frac{\Phi_p\left(u_n\right)}{\|u_n\|^p_{W^{1,p}(\Omega)}} = \frac{1}{p} - \int_{\partial\Omega} \frac{F\left(x, u_n\left(x\right)\right)}{\|u_n\|^p_{W^{1,p}(\Omega)}} \partial\sigma - \int_{\partial\Omega} h \frac{v_n\left(x\right)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \partial\sigma,$$

so

$$\begin{array}{rcl} \frac{1}{p} & = & \int_{\partial\Omega} \frac{F(x,u_n(x))}{\|u_n\|^p_{W^{1,p}(\Omega)}} \partial\sigma + \int_{\partial\Omega} h \frac{v_n(x)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \partial\sigma + \frac{\Phi_p(u_n)}{\|u_n\|^p_{W^{1,p}(\Omega)}} \\ & \geq & \int_{\partial\Omega} \left(L\left(x\right) - \varepsilon\right) \frac{\left|v_n(x)\right|^p}{p} \partial\sigma - \int_{\partial\Omega} \frac{d_\varepsilon(x)}{\|u_n\|^p_{W^{1,p}(\Omega)}} \partial\sigma \\ & + & \int_{\partial\Omega} h \frac{v_n(x)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \partial\sigma + \frac{\Phi_p(u_n)}{\|u_n\|^p_{W^{1,p}(\Omega)}}, \end{array}$$

passing to the limit one finds

$$\frac{1}{p} \geq \int_{\partial\Omega} \left(L\left(x\right) - \varepsilon\right) \frac{\left|v\left(x\right)\right|^p}{p} \partial\sigma,$$

this's true for all $\varepsilon > 0$, one deducts that $1 \geq \int_{\partial\Omega} L\left(x\right) \left|v\left(x\right)\right|^p \partial\sigma$. In addition, one has

$$1 = \|v\|^p_{W^{1,p}(\Omega)} = \int_{\partial\Omega} m\left(x\right) \left|v\right|^p \partial\sigma = \lambda_1 \int_{\partial\Omega} \left|v\right|^p \partial\sigma,$$

then $\int_{\partial\Omega} \left(L\left(x\right) - \lambda_1\right) \left|v\right|^p \partial\sigma \leq 0$, witch contradicts the hypothesis $\lambda_1 \lneqq L\left(x\right)$ a.e in $\partial\Omega$. We must here show that $m\left(x\right) < \lambda_2$ a.e in $\partial\Omega$. In the opposite case, by the theorem 2.3, one has v an eigenfunction associated to λ_2, i.e $1 = \|v\|^p_{W^{1,p}(\Omega)} = \lambda_2 \int_{\partial\Omega} \left|v\right|^p \partial\sigma$, $\left|\partial\Omega^{\pm}\left(v\right)\right| \geq \left(\frac{S_{P*}}{\lambda_2}\right)^\eta$ and by (2.5), one has

$$\int_{\partial\Omega} K\left(x\right) \left|v\left(x\right)\right|^p \partial\sigma \geq 1 = \lambda_2 \int_{\partial\Omega} \left|v\right|^p \partial\sigma,$$

so $K(x) = \lambda_2$ a.e in $\{x \in \partial\Omega : v(x) \neq 0\}$, then

$$\begin{array}{rcl} |\{x \in \partial\Omega : K(x) = \lambda_2\}| & \geq & |\{x \in \partial\Omega : v(x) \neq 0\}| \\ & \geq & |\partial\Omega^+(v) \cup \partial\Omega^-(v)| \\ & \geq & 2\left(\frac{S_{P^*}}{\lambda_2}\right)^\eta, \end{array}$$

this contradicts (2.3), consequently $\lambda_1 \lneqq m(x) < \lambda_2$ a.e in $\partial\Omega$. □

Let us return to the demonstration of the theorem 2.4. The result of theorem 2.2 assures that

$$1 = \lambda_2(\lambda_2) < \lambda_2(m),$$

then 1 is an eigenvalue of the problem $P(m)$ strictly between $\lambda_1(m)$ and $\lambda_2(m)$, absurd. Finally Φ is (PS).

ii) For applying the theorem 2.6, one constructs two sets E and Q_0 satisfying (P1) and (P2).

Let $E = \left\{u \in W^{1,p}(\Omega) : \lambda_2(K)\int_{\partial\Omega} K(x)|u|^p \partial\sigma \leq \|u\|^p_{W^{1,p}(\Omega)}\right\}$, by (2.5) and the propriety $\int_{\partial\Omega}|u|^p \partial\sigma \leq \frac{1}{\lambda_1(1)}\|u\|^p_{W^{1,p}(\Omega)} = \frac{1}{\lambda_1(1)}$, one has for $u \in E$

$$\begin{array}{rcl} \Phi(u) & = & \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \int_{\partial\Omega} F(x,u)\partial\sigma - \int_{\partial\Omega} hu\partial\sigma \\ & \geq & \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \int_{\partial\Omega}(K(x)+\varepsilon)\frac{|u|^p}{p}\partial\sigma - \int_{\partial\Omega}(d_\varepsilon(x)+hu)\partial\sigma \\ & \geq & \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{1}{p\lambda_2(K)}\|u\|^p_{W^{1,p}(\Omega)} - \int_{\partial\Omega}\frac{\varepsilon}{p}|u|^p - \int_{\partial\Omega}(d_\varepsilon(x)+hu) \\ & \geq & \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}\left(1 - \frac{1}{\lambda_2(K)} - \frac{\varepsilon}{\lambda_1(1)}\right) - \int_{\partial\Omega}(d_\varepsilon(x)+hu)\partial\sigma. \end{array}$$

Indeed one knows that $K(x) \leq \lambda_2$ a.e in $\partial\Omega$ implies that $\lambda_2(K) \geq 1$, moreover if $\lambda_2(K) = 1$ then 1 is an eigenvalue for the problem $P(K)$ with $\lambda_1 \lneqq K(x) \leq \lambda_2$ a.e in $\partial\Omega$, and from theorem 2.3,

$$meas_\sigma\{x \in \partial\Omega : K(x) = \lambda_2\} \geq 2\left(\frac{S_{P^*}}{\|K\|_{L^\infty(\partial\Omega)}}\right)^\eta \geq 2\left(\frac{S_{P^*}}{\lambda_2}\right)^\eta,$$

this is in contradiction with the hypothesis (2.3), so $1 - \frac{1}{\lambda_2(K)} > 0$. Then for ε small enough $1 - \frac{1}{\lambda_2(K)} - \frac{\varepsilon}{\lambda_1(1)} > 0$, consequently Φ is coercive in E. Let ξ be an eigenfunction associated to $\lambda_1(L)$ with $\|\xi\|_{W^{1,p}(\Omega)} = 1$, one shows that $\lim_{|t|\to+\infty}\Phi(t\xi) = -\infty$. From (2.5) one finds

$$\begin{array}{rcl} \Phi(t\xi) & = & \frac{1}{p}|t|^p - \int_{\partial\Omega} F(x,t\xi)\partial\sigma - t\int_{\partial\Omega} h\xi\partial\sigma \\ & \leq & \frac{1}{p}|t|^p - \int_{\partial\Omega}(L(x)-\varepsilon)\frac{|t\xi|^p}{p}\partial\sigma + \int_{\partial\Omega}(d_\varepsilon(x)-ht\xi)\partial\sigma \\ & \leq & \frac{1}{p}|t|^p - \frac{1}{p\lambda_1(L)}\|u\|^p_{W^{1,p}(\Omega)} + \int_{\partial\Omega}\frac{\varepsilon}{p}|t\xi|^p\partial\sigma + \int_{\partial\Omega}(d_\varepsilon(x)-ht\xi)\partial\sigma \\ & \leq & \frac{1}{p}|t|^p\left(1 - \frac{1}{\lambda_1(L)} + \frac{\varepsilon}{\lambda_1(1)}\right) + \int_{\partial\Omega}(d_\varepsilon(x)-ht\xi)\partial\sigma, \end{array}$$

and since $\lambda_1 \lneqq L(x)$ a.e in $\partial\Omega$, one obtains $1 > \lambda_1(L)$, then for ε small enough $1 - \frac{1}{\lambda_1(L)} + \frac{\varepsilon}{\lambda_1(1)} < 0$, thus the result is proven. We pose $\beta := \inf_E \Phi$, it's finite, and $\lim_{|t| \to +\infty} \Phi(t\xi) = -\infty$ implies that there exists t' sufficiently big such that

$$\alpha := \max\left(\Phi(t'\xi), \Phi(-t'\xi)\right) < \beta.$$

Take the compact $Q_0 = \{-t'\xi, t'\xi\}$, it's clear that $\theta(Q_0) = 1$, and (P2) is satisfied. Let Q be a symmetric compact in $W^{1,p}(\Omega)$ such that $\gamma(Q) \geq 2$, by the definition of $\lambda_2(K)$ one has

$$\min_{w \in Q} \frac{\int_{\partial\Omega} K(x)|w|^p \partial\sigma}{\|w\|^p_{W^{1,p}(\Omega)}} \leq \frac{1}{\lambda_2(K)},$$

thus there exists $w_0 \in Q$ such that $\frac{\int_{\partial\Omega} K(x)|w_0|^p \partial\sigma}{\|w_0\|^p_{W^{1,p}(\Omega)}} \leq \frac{1}{\lambda_2(K)}$ i.e $w_0 \in E$, and (P1) is satisfied. One deducts that the value

$$c := \inf_{h \in \Gamma} \max_{u \in h(\overline{D})} \Phi \geq \beta$$

is a critical value for the functional Φ, with $\Gamma = \left\{h \in C\left(\overline{D}, X \setminus \{0\}\right) : h = id \text{ in } Q_0\right\}$ and $D = co(Q_0)$. The proof is completed.

3.5. Proof of theorem 2.5. The same functional Φ is taken, it's C^1 and weakly lower semi-continuous. According to the theorem 2.7, it remains to prove that it's coercive in $W^{1,p}(\Omega)$. Indeed, for $u \in W^{1,p}(\Omega)$ one has

$$\begin{aligned} \Phi(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \int_{\partial\Omega} F(x,u)\,\partial\sigma - \int_{\partial\Omega} hu\partial\sigma \\ &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \int_{\partial\Omega} (K(x) + \varepsilon)\frac{|u|^p}{p}\partial\sigma - \int_{\partial\Omega} (d_\varepsilon(x) + hu)\,\partial\sigma, \end{aligned}$$

and

$$\int_{\partial\Omega} K(x)|u|^p \partial\sigma \leq \frac{1}{\lambda_1(K)}\|u\|^p_{W^{1,p}(\Omega)},$$

then

$$\Phi(u) \geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}\left(1 - \frac{1}{\lambda_1(K)} - \frac{\varepsilon}{\lambda_1(1)}\right) - \int_{\partial\Omega} (d_\varepsilon(x) + hu)\,\partial\sigma.$$

From $K(x) \lneqq \lambda_1$ a.e in $\partial\Omega$, and the theorem 2.1, one has $1 - \frac{1}{\lambda_1(K)} > 0$, then for ε small enough $\left(1 - \frac{1}{\lambda_1(K)} - \frac{\varepsilon}{\lambda_1(1)}\right) > 0$, thus Φ is coercive in $W^{1,p}(\Omega)$. Finally, Φ admits a critical point witch's a solution of the problem (2.1).

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