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Existence for an elliptic system with nonlinear boundary conditions

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ABSTRACT: In this paper we prove the existence of a weak solution to the following system

$$\begin{cases} \triangle_p u = \triangle_q v = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f(x,u) - (\alpha+1)K(x)|u|^{\alpha-1}u|v|^{\beta+1} + f_1 & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = g(x,v) - (\beta+1)K(x)|v|^{\beta-1}v|u|^{\alpha+1} + g_1 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, f_1, g_1, f, g and K are functions that satisfy some conditions.

Key Words: Steklov problems, weights, elliptic systems, nonlinear boundary conditions.

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1. Introduction

Consider the system with nonlinear boundary conditions

$$\begin{cases} \triangle_p u = \triangle_q v = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f(x,u) - (\alpha+1)K(x)|u|^{\alpha-1}u|v|^{\beta+1} + f_1 & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = g(x,v) - (\beta+1)K(x)|v|^{\beta-1}v|u|^{\alpha+1} + g_1 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Ω will be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz continuous boundary, $1 < p < \infty$, $1 < q < \infty$ and suppose the following conditions:

$$\alpha \geq 0,\ \beta \geq 0,\ \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \quad K \in L^\infty(\partial\Omega), \quad K \geq 0.$$

$$f_1 \in L^r(\partial\Omega),\ r = \frac{p\overline{p}}{p\overline{p} - \overline{p} + 1},\ \frac{N-1}{p-1} < \overline{p} < \infty \text{ if } p < N \text{ and } \overline{p} \geq 1 \text{ if } p \geq N.$$

$$g_1 \in L^{\overline{r}}(\partial\Omega),\ \overline{r} = \frac{q\overline{q}}{q\overline{q} - \overline{q} + 1},\ \frac{N-1}{q-1} < \overline{q} < \infty \text{ if } q < N \text{ and } \overline{q} \geq 1 \text{ if } q \geq N.$$

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$f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that verify:
$(f_{a,b})$: There exist $a > 0$ and a function $b \in L^r(\partial\Omega)$ such that,

$$|f(x,s)| \leq a|s|^{p-1} + b(x) \text{ a.e. } x \in \partial\Omega \text{ and for all } s \in \mathbb{R}.$$

$\limsup_{|s|\to+\infty} \frac{pF(x,s)}{|s|^p} := m(x) \in L^{\overline{p}}(\partial\Omega)$ with $F(x,s) = \int_0^s f(x,t)dt$.
$(g_{c,d})$: There exist $c > 0$ and a function $d \in L^{\overline{r}}(\partial\Omega)$ such that,

$$|g(x,s)| \leq c|s|^{q-1} + d(x) \text{ a.e. } x \in \partial\Omega \text{ and for all } s \in \mathbb{R}.$$

$\limsup_{|s|\to+\infty} \frac{qG(x,s)}{|s|^q} := n(x) \in L^{\overline{q}}(\partial\Omega)$ with $G(x,s) = \int_0^s g(x,t)dt$.
We say that $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int_{\partial\Omega} f(x,u)\varphi d\sigma - (\alpha+1)\int_{\partial\Omega} K(x)|u|^{\alpha-1}u|v|^{\beta+1}\varphi d\sigma + \int_{\partial\Omega} f_1\varphi d\sigma,$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi dx = \int_{\partial\Omega} g(x,v)\psi d\sigma - (\beta+1)\int_{\partial\Omega} K(x)|u|^{\alpha+1}|v|^{\beta-1}v\psi d\sigma + \int_{\partial\Omega} g_1\psi d\sigma.$$

for all $(\varphi,\psi) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure.

Existence results for nonlinear elliptic systems when the nonlinear term appears as a source in the equation complemented with Dirichlet boundary conditions have been studied by various authors; we cite the works [1,3,4]. For the nonlinear boundary condition, the authors in [2] proved the existence of nontrivial solutions to the system

$$\begin{cases} \triangle_p u = |u|^{p-2}u, \triangle_q v = |v|^{p-2}v & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = F_u(x,u,v), |\nabla v|^{q-2}\frac{\partial v}{\partial\nu} = F_v(x,u,v) & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $(F_u; F_v)$ is the gradient of some positive potential $F : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The proofs are done under suitable assumptions on the potential F, and based on variational arguments.

Our purpose in the present paper is to show that the problem (1.1) admits at least a solution $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, we also give a special case of the problem (1.1) (see Corollary 3.3A). Our proofs are based on variational arguments.

2. Preliminaries

In this section, we collect some results relative to the eigenvalue problem

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = \lambda m(x)|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the weight m is assumed to lie in $M_{\bar{p}} := \{m \in L^{\bar{p}}(\partial\Omega); m^+ \not\equiv 0 \text{ and } \int_{\partial\Omega} m d\sigma < 0\}$.

O. Torné in [5] showed, by using infinite dimensional Ljusternik–Schnirelman theory, that the problem (2.1) admits a sequence of eigenvalues

$$\lambda_k(m,p) := \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{1}{p}\int_\Omega |\nabla u|^p dx,$$

where

$$\Gamma_k := \{C \subset S; C \text{ is symmetric, compact and } \gamma(C) \geq k\}$$

with

$$S := \left\{u \in W^{1,p}(\Omega); \frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma = 1\right\},$$

$\gamma(C)$ is the Krasnoselski genus of C, and let

$$\lambda_1(m,p) = \inf\left\{\frac{1}{p}\int_\Omega |\nabla u|^p; u \in W^{1,p}(\Omega) \text{ and } \frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma = 1\right\}.$$

Theorem 2.1 *([5]) Assume $m \in M_{\bar{p}}$. Then $\lambda_k(m,p)$ is a nondecreasing and unbounded sequence of positive eigenvalues of the problem (2.1). Moreover $\lambda_1(m,p) > 0$ is the first positive eigenvalue of (2.1). Moreover $\lambda_1(m,p)$ is simple, isolated and it is the only nonzero eigenvalue associated to an eigenfunction of definite sign.*

Remark 2.2 This theorem is proved in [5] by applying infinite dimensional Ljusternik–Schnirelman theory for existence of the sequence $\lambda_k(m,p)$ and Picone's identity for simplicity of the first eigenvalue.

3. Existence of solution for a system Steklov problem

In the whole continuation, we note by $\lambda_1(m,p)$ (resp. $\lambda_1(n,q)$) the first eigenvalue of the problem (2.1) for the integer p and the weight m (resp. the integer q and the weight n). We also note

$$M_{\bar{p}} := \{m \in L^{\bar{p}}(\partial\Omega); m^+ \not\equiv 0 \text{ and } \int_{\partial\Omega} m d\sigma < 0\},$$

$$M_{\bar{q}} := \{m \in L^{\bar{q}}(\partial\Omega); m^+ \not\equiv 0 \text{ and } \int_{\partial\Omega} m d\sigma < 0\}.$$

Theorem 3.1 If $m \in M_{\bar{p}}$ and $n \in M_{\bar{q}}$, then the problem (1.1) admits at least a solution for $\lambda_1(m,p) > 1$ and $\lambda_1(n,q) > 1$.

Remark 3.2 We can have $\lambda_1(m,p) > 1$, since $\lambda_1(m,p)$ is homogeneous respect to the weight in the sense where

$$\lambda_1(\alpha m,p) = \frac{\lambda_1(m,p)}{\alpha} \qquad \forall\alpha > 0.$$

Consider the space $W = W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ equipped with the norm

$$||w|| = ||u||_{1,p} + ||v||_{1,q}, \text{ for } w = (u,v) \in W,$$

where

$$||u||_{1,p} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\right)^{\frac{1}{p}} \text{ and } ||v||_{1,q} = \left(\int_\Omega |\nabla v|^q dx + \int_\Omega |v|^q dx\right)^{\frac{1}{q}}.$$

Let the energy functional $\Phi : W \to \mathbb{R}$ such that

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_{\partial\Omega} F(x,u)d\sigma + \frac{1}{q}\int_\Omega |\nabla v|^q dx - \int_{\partial\Omega} G(x,v)d\sigma$$
$$+ \int_{\partial\Omega} K(x)|u|^{\alpha+1}|v|^{\beta+1}d\sigma - \int_{\partial\Omega} f_1 u d\sigma - \int_{\partial\Omega} g_1 v d\sigma.$$

Remark 3.3 *The conditions* $\limsup_{|s|\to+\infty} \frac{pF(x,s)}{|s|^p} := m(x)$ *imply that for all* $\varepsilon > 0$*, there exists* $d_\varepsilon \in L^r(\partial\Omega)$ *such that a.e.* $x \in \partial\Omega$ *and for all* $s \in \mathbb{R}$*, we have*

$$F(x,s) \le (m(x)+\varepsilon)\frac{|s|^p}{p} + d_\varepsilon(x).$$

$\limsup_{|s|\to+\infty} \frac{qG(x,s)}{|s|^q} := n(x)$ *imply that for all* $\varepsilon > 0$*, there exists* $d'_\varepsilon \in L^{\overline{r}}(\partial\Omega)$ *such that a.e.* $x \in \partial\Omega$ *and for all* $s \in \mathbb{R}$*, we have*

$$G(x,s) \le (n(x)+\varepsilon)\frac{|s|^p}{p} + d'_\varepsilon(x).$$

Proposition 3.1 $m \to \lambda_1(m,p)$ *is continuous from* $M_{\bar{p}}$ *into* $\mathbb{R}$.

Proof: Let m_k, $m \in M_{\bar{p}}$ such that $m_k \to m$ in $L^{\bar{p}}(\partial\Omega)$. By definition of $\lambda_1(m,p)$, for $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,p}(\Omega)$ verifying $\frac{1}{p}\int_{\partial\Omega} m|u_\varepsilon|^p d\sigma = 1$ and $\frac{1}{p}\int_\Omega |\nabla u_\varepsilon|^p dx \le \lambda_1(m,p) + \varepsilon$.
Since $(m,u) \to \frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma$ is continuous in its two arguments (m,u), we deduce for k sufficiently large,

$$\frac{1}{p}\int_\Omega \left|\nabla \frac{u_\varepsilon}{(\frac{1}{p}\int_{\partial\Omega} m_k|u_\varepsilon|^p d\sigma)^{1/p}}\right|^p \le \lambda_1(m,p) + \varepsilon.$$

Consequently

$$\lambda_1(m_k,p) \le \lambda_1(m,p) + \varepsilon.$$

Thus,

$$\limsup_{k\to+\infty} \lambda_1(m_k,p) \le \lambda_1(m,p) + \varepsilon.$$

As ε is arbitrary,

$$\limsup_{k\to+\infty} \lambda_1(m_k,p) \le \lambda_1(m,p).$$

Now we must show that $\lambda_1(m,p) \le \liminf_{k\to+\infty} \lambda_1(m_k,p)$. Suppose by contradiction that

$$\lambda_1(m,p) > \liminf_{k\to+\infty} \lambda_1(m_k,p) = \lambda.$$

Let $u_k \in W^{1,p}(\Omega)$ such that $\frac{1}{p}\int_\Omega m_k|u_k|^p d\sigma = 1$ be a solution of the problem

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda_1(m_k(x),p)m_k(x)|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We distinguish two cases.
First case: $||u_k||_{1,p} \to +\infty$, put $v_k = \frac{u_k}{||u_k||_{1,p}}$, for a subsequence still denoted by (v_k) there exists $v \in W^{1,p}(\Omega)$ such that $v_k \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and $v_k \to v$ strongly in $L^p(\partial\Omega)$. Since u_k is a solution of (3.1) and $\frac{1}{p}\int_{\partial\Omega} m_k|u_k|^p d\sigma = 1$, then $\frac{1}{p}\int_\Omega |\nabla u_k|^p dx = \lambda_1(m_k,p)$. Dividing by $||u_k||^p_{1,p}$ and passing to the limit, we have $\frac{1}{p}\int_\Omega |\nabla v_k|^p dx \to 0$. Thus $v_k \to v$ strongly in $W^{1,p}(\Omega)$ and $v = cst \neq 0$. But $\frac{1}{p}\int_{\partial\Omega} m_k|v_k|^p d\sigma = \frac{1}{||u_k||^p_{1,p}}$, passing to limit, we obtain $\frac{1}{p}|cst|^p \int_{\partial\Omega} m d\sigma = 0$, this contradicts $\int_{\partial\Omega} m d\sigma < 0$ and $cst \neq 0$.
Seconde case: (u_k) is bounded, for a subsequence still denoted (u_k), there exists $u \in W^{1,p}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, $u_k \to u$ strongly in $L^p(\Omega)$ and $u_k \to u$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$. Since u_k is a solution of (3.1), then

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla\varphi dx = \lambda_1(m_k,p)\int_{\partial\Omega} m_k|u_k|^{p-2}u_k\varphi d\sigma \quad \forall\varphi \in W^{1,p}(\Omega).$$

Now taking $\varphi = u_k - u$, we have

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k-u) dx = \lambda_1(m_k,p)\int_{\partial\Omega} m_k|u_k|^{p-2}u_k(u_k-u) d\sigma.$$

Passing to the limit, we obtain

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k-u) dx \to 0 \text{ as } k\to+\infty.$$

On the other hand

$$\int_\Omega |u_k|^{p-2}u_k(u_k-u) dx \to 0 \text{ as } k\to+\infty.$$

It then follows the (S^+) property that $u_k \to u$ strongly in $W^{1,p}(\Omega)$. In addition $u \neq 0$ (since $\frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma = 1$). Thus λ is an eigenvalue of the problem (1.1). Since $\int_{\partial\Omega} m d\sigma < 0$, we deduce $0 < \lambda$. Consequently, we have $0 < \lambda < \lambda_1(m,p)$, this contradicts the Theorem 2.1. Finally, we have

$$\limsup_{k\to+\infty} \lambda_1(m_k,p) \le \lambda_1(m,p) \le \liminf_{k\to+\infty} \lambda_1(m_k,p), \text{ so } \lambda_1(m_k,p) \to \lambda_1(m,p).$$

□

Lemma 3.1 *If $m \in M_{\bar{p}}$ and $n \in M_{\bar{q}}$, then the functional Φ is coercive for $\lambda_1(m,p) > 1$ and $\lambda_1(n,q) > 1$.*

Proof: Suppose by contradiction that there exist a sequence $w_n \in W$ and $c \geq 0$ with $w_n = (u_n, v_n)$ such that $||w_n|| \to +\infty$ and $|\Phi(w_n)| \leq c$. Put $k_n = ||w_n||$, $\tilde{u_n} = \frac{u_n}{k_n}$ and $\tilde{v_n} = \frac{v_n}{k_n}$. $c \geq |\Phi(w_n)|$ implies that

$$\begin{aligned} c \geq \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \int_{\partial\Omega} F(x,u_n)d\sigma + \frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \int_{\partial\Omega} G(x,u_n)d\sigma \\ + \int_{\partial\Omega} K(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}d\sigma - \int_{\partial\Omega} f_1 u_n d\sigma - \int_{\partial\Omega} g_1 v_n d\sigma. \end{aligned} \tag{3.2}$$

Since $\int_{\partial\Omega} K(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}d\sigma \geq 0$, by the Remark (3.3) we have

$$\begin{aligned} c \geq \frac{1}{p}\int_\Omega |\nabla u_n|^p dx + \frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \frac{1}{p}\int_{\partial\Omega}(m(x)+\varepsilon)|u_n|^p d\sigma - \frac{1}{q}\int_{\partial\Omega}(n(x)+\varepsilon)|v_n|^q d\sigma \\ - \int_{\partial\Omega} d_\varepsilon(x)d\sigma - \int_{\partial\Omega} d'_\varepsilon(x)d\sigma - \int_{\partial\Omega} f_1 u_n d\sigma - \int_{\partial\Omega} g_1 v_n d\sigma. \end{aligned} \tag{3.3}$$

Let $\varepsilon > 0$ such that $\lambda_1(m+\varepsilon,p) > 1$ and $\lambda_1(n+\varepsilon,q) > 1$ (the continuity of $m \to \lambda_1(m,p)$ is used here, see Proposition 3.1). Thus

$$\begin{aligned} c \geq \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{1}{p}\int_{\partial\Omega}(m(x)+\varepsilon)|u_n|^p d\sigma - \int_{\partial\Omega} d_\varepsilon(x)d\sigma - \int_{\partial\Omega} d'_\varepsilon(x)d\sigma \\ - \int_{\partial\Omega} f_1 u_n d\sigma - \int_{\partial\Omega} g_1 v_n d\sigma, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} c \geq \frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \frac{1}{q}\int_{\partial\Omega}(n(x)+\varepsilon)|v_n|^q d\sigma - \int_{\partial\Omega} d_\varepsilon(x)d\sigma - \int_{\partial\Omega} d'_\varepsilon(x)d\sigma \\ - \int_{\partial\Omega} f_1 u_n d\sigma - \int_{\partial\Omega} g_1 v_n d\sigma. \end{aligned} \tag{3.5}$$

Therefore

$$\begin{aligned} c \geq \left(1 - \frac{1}{\lambda_1(m+\varepsilon,p)}\right)\frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \int_{\partial\Omega} d_\varepsilon(x)d\sigma - \int_{\partial\Omega} d'_\varepsilon(x)d\sigma \\ - \int_{\partial\Omega} f_1 u_n d\sigma - \int_{\partial\Omega} g_1 v_n d\sigma, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} c \geq \left(1 - \frac{1}{\lambda_1(n+\varepsilon,q)}\right)\frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \int_{\partial\Omega} d_\varepsilon(x)d\sigma - \int_{\partial\Omega} d'_\varepsilon(x)d\sigma \\ - \int_{\partial\Omega} f_1 u_n d\sigma - \int_{\partial\Omega} g_1 v_n d\sigma. \end{aligned} \tag{3.7}$$

Dividing (3.6) and (3.7) respectively by k_n^p and k_n^q, we obtain

$$\begin{aligned}\frac{c}{k_n^p} \geq & \left(1-\frac{1}{\lambda_1(m+\varepsilon,p)}\right)\frac{1}{p}\int_\Omega |\nabla\tilde{u}_n|^p dx \\ & -\frac{1}{k_n^p}\left[\int_{\partial\Omega} d_\varepsilon(x)d\sigma+\int_{\partial\Omega} d'_\varepsilon(x)d\sigma\int_{\partial\Omega} f_1 u_n d\sigma+\int_{\partial\Omega} g_1 v_n d\sigma\right],\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{c}{k_n^q} \geq & \left(1-\frac{1}{\lambda_1(n+\varepsilon,q)}\right)\frac{1}{q}\int_\Omega |\nabla\tilde{v}_n|^q dx \\ & -\frac{1}{k_n^q}\left[\int_{\partial\Omega} d_\varepsilon(x)d\sigma+\int_{\partial\Omega} d'_\varepsilon(x)d\sigma\int_{\partial\Omega} f_1 u_n d\sigma+\int_{\partial\Omega} g_1 v_n d\sigma\right].\end{aligned} \tag{3.9}$$

Since $\tilde{u_n}$ is a bounded, for a further subsequence still denoted by $\tilde{u_n} \rightharpoonup \tilde{u}$ weakly in $W^{1,p}(\Omega)$ and $\tilde{u_n} \to \tilde{u}$ strongly in $L^p(\Omega)$, on the other hand we have

$$\int_\Omega |\nabla\tilde{u}|^p dx+\int_\Omega |\tilde{u}|^p dx \leq \liminf_{n\to+\infty}\left(\int_\Omega |\nabla\tilde{u_n}|^p dx+\int_\Omega |\tilde{u_n}|^p dx\right).$$

In (3.8), passing to the limit we obtain $0=\int_\Omega |\nabla\tilde{u}|^p dx$, thus $\tilde{u}=c_1=cst$ and $||\tilde{u_n}||_{1,p} \to ||\tilde{u}||_{1,p}$. Since $W^{1,p}(\Omega)$ is uniformly convex and reflexive, $\tilde{u_n} \to cst = c_1$ strongly in $W^{1,p}(\Omega)$. (By a similar argument we show that $\tilde{v_n} \to cst = c_2$ strongly in $W^{1,p}(\Omega)$). Dividing (3.4), (3.5) respectively by k_n^p and k_n^q and passing to the limit, we have

$$0 \geq -\frac{|c_1|^p}{p}\int_{\partial\Omega}(m(x)+\varepsilon)d\sigma \text{ and } 0 \geq -\frac{|c_2|^q}{q}\int_{\partial\Omega}(n(x)+\varepsilon)d\sigma.$$

Since ε is arbitrary, we obtain

$$\frac{|c_1|^p}{p}\int_{\partial\Omega} m(x)d\sigma \geq 0 \text{ and } \frac{|c_2|^q}{q}\int_{\partial\Omega} n(x)d\sigma \geq 0.$$

Since $\int_{\partial\Omega} m(x)d\sigma<0$ and $\int_{\partial\Omega} n(x)d\sigma<0$, then $c_1=c_2=0$, consequently $\|\tilde{w}_n\| \to 0$, where $\tilde{w}_n := (\tilde{u}_n,\tilde{v}_n)$. This contradicts $\|\tilde{w}_n\|=1$. Finally Φ is a coercive. ☐

Lemma 3.2 *If $m \in M_{\overline{p}}$ and $n \in M_{\overline{q}}$, then the energy functional Φ is a weakly lower semicontinuous.*

Proof: It suffices to see that the trace mapping $W \to L^{\frac{p\overline{p}}{\overline{p}-1}}(\partial\Omega) \times L^{\frac{q\overline{q}}{\overline{q}-1}}(\partial\Omega)$ is compact. ☐

Proof: [Proof of Theorem 2.1.] By Lemma 3.2, Φ is weakly lower semicontinuous and by Lemma 3.1, Φ is coercive. Φ is continuously differentiable. The proof is complete. ☐

Now we can give a special case of Theorem 2.1.

Corollary 3.3A *Suppose that $m \in M_{\overline{p}}$ and $n \in M_{\overline{q}}$. If $\lambda_1(m,p)>1$, $\lambda_1(n,q)>1$ then the problem (1.1) admits at least a solution for $f(x,u)=m|u|^{p-2}u$ and $g(x,v)=n|v|^{q-2}v$.*

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