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On Semiprime Rings with Generalized Derivations

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ABSTRACT: Let R be a ring and F and G be generalized derivations of R with associated derivations d and g respectively. In the present paper, we shall investigate the commutativity of semiprime ring R admitting generalized derivations F and G satisfying any one of the properties: $(i) F(x)x = xG(x)$, (ii) $[F(x), d(y)] = [x, y]$, (iii) $F([x, y]) = [d(x), F(y)]$, (iv) $d(x)F(y) = xy$, (v) $F(x^2) = x^2$, (vi) $[F(x), y] = [x, G(y)]$, (vii) $F([x, y]) = [F(x), y] + [d(y), x]$ and $(viii)$ $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all x, y in some appropriate subset of R.

Key Words: Ideal, semiprime ring, derivation, generalized derivation.

Contents

1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for commutator $xy-yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy+yx$. We shall make extensive use of the following basic identities without any specific mention : $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = y[x, z] + [x, y]z$, $xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$ and $(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z]$.

Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$, and R is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, for a fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [x, a]$ is a derivation called an inner derivation. Let S be a non-empty subset of R. A mapping $f : R \longrightarrow R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$.

An additive mapping $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \text{ for all } x, y \in R.$$

This observation leads to the following definition, given in [10] and [12]; an additive mapping $F : R \longrightarrow R$ is called a generalized derivation with associated derivation d if

$$F(xy) = F(x)y + xd(y) \text{ holds for all } x, y \in R.$$

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Familiar examples of generalized derivations are derivations and generalized inner derivations and the later includes left multiplier i.e., an additive map $F : R \longrightarrow R$ satisfying $F(xy) = F(x)y$ for all $x, y \in R$. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is fixed element of R and d a derivation of R, is a generalized derivation; and if R has 1, all generalized derivations have this form.

Over the last three decades, several authors have proved commutativity theorems for prime or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on some appropriate subsets of R (see [1-7], [10], [11], [13] and [15] where further references can be found). The purpose of this paper is to prove some results which are of independent interest and related to generalized derivations on semiprime rings. In fact, our results extend some known theorems for derivations to generalized derivations in the setting of semiprime rings viz., Theorem 3.1 of [2] and Theorems 2.9-2.12 of [3].

2. Main Results

We begin our discussion with the following theorem which extends some known results obtained in [9] and [14].

Theorem 2.1 *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g respectively. If $F(x)x = xG(x)$ for all $x \in I$ or if $F(x)x + xG(x) = 0$ for all $x \in I$, then R contains a nonzero central ideal.*

We start with the following known results which are essentially proved in [6], [7] and [8] respectively.

Lemma 2.1 *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d. If $F(xoy) - (xoy) = 0$ for all $x, y \in I$ or if $F(xoy) + (xoy) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Lemma 2.2 *Let R be a semiprime ring and I be a nonzero left ideal of R. If R admits a nonzero derivation d such that $[x, d(x)]$ is central for all $x \in I$, then R contains a nonzero central ideal.*

Lemma 2.3 *Let R be a 2-torsion free semiprime ring and L be a left ideal of R. If $a, b \in R$ and $axb + bxa = 0$ for all $x \in I$, then $axb = bxa = 0$ for all $x \in I$.*

Proof of Theorem 2.1: By hypothesis, we have

$$F(x)x = xG(x) \text{ for all } x \in I.$$

On linearizing the above relation we find that

$$F(x)y + F(y)x = xG(y) + yG(x) \text{ for all } x, y \in I. \tag{2.1}$$

Replace x by xy in (2.1), to get

$$F(x)y^2 + xd(y)y + F(y)xy = xyG(y) + yG(x)y + yxg(y) \text{ for all } x, y \in I. \quad (2.2)$$

Right multiplication by y to the relation (2.1) yields that

$$F(x)y^2 + F(y)xy = xG(y)y + yG(x)y \text{ for all } x, y \in I. \quad (2.3)$$

Combining (2.2) and (2.3), we obtain

$$xd(y)y = yxg(y) + x[y, G(y)] \text{ for all } x, y \in I. \quad (2.4)$$

Now, replacing x by rx in (2.4), we get

$$rxd(y)y = yrxg(y) + rx[y, G(y)] \text{ for all } x, y \in I, r \in R. \quad (2.5)$$

Left multiplying to (2.4) by r, we arrive at

$$rxd(y)y = ryxg(y) + rx[y, G(y)] \text{ for all } x, y \in I, r \in R. \quad (2.6)$$

From (2.5) and (2.6), we get $[y, r]xg(y) = 0$ and hence $[y, g(y)]xg(y) = 0$ for all $x, y \in I$. That is, $I[y, g(y)]RI[y, g(y)] = \{0\}$ for all $y \in I$. Then by the semiprimeness of R, we find that $I[y, g(y)] = 0$ for all $y \in I$. This yields that $[y, r][y, g(y)] = 0$ for all $y \in I$ and $r \in R$. Now replacing r by $g(y)r$, we obtain $[y, g(y)]R[y, g(y)] = \{0\}$ for all $y \in I$. Again by the semiprimeness of R, we find that $[y, g(y)] = 0$ for all $y \in I$. Hence by Lemma 2.2, we get the required result.

Further, if $F(x)x + xG(x) = 0$ for all $x \in I$, then using the same techniques as used above with necessary variations we get the required result. This completes the proof of our theorem.

Following is an immediate corollary of the above theorem.

Corollary 2.1A *([2, Theorem 3.1]). Let R be a semiprime ring. Let d and g be derivations of R such that at least one of them is nonzero. If $d(x)x = xg(x)$ for all $x \in R$, then R contains a nonzero central ideal.*

Proceeding on the same lines with necessary variations and taking $G = F$ or $G = -F$ in Theorem 2.1, we get the following:

Corollary 2.1B *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $[F(x), x] = 0$ for all $x \in I$ or if $F(x) \circ x = 0$ for all $x \in I$. Then R contains a nonzero central ideal.*

Theorem 2.2 *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $F(x^2) = x^2$ for all $x \in I$ or if $F(x^2) + x^2 = 0$ for all $x \in I$. Then R contains a nonzero central ideal.*

Proof: We have

$$F(x^2) = x^2 \text{ for all } x \in I.$$

Replacing x by $x + y$ in the above relation, we get

$$F(x^2 + y^2 + xy + yx) = x^2 + y^2 + xy + yx \text{ for all } x, y \in I. \quad (2.7)$$

Using the given hypothesis in (2.7), we obtain

$$F(xy + yx) = xy + yx \text{ for all } x, y \in I. \quad (2.8)$$

This can be written as $F(x \circ y) = x \circ y$ for all $x, y \in I$ and hence by Lemma 2.1, we get the required result. □

Further, if $F(x^2) + x^2 = 0$ for all $x \in I$, then we find that $F(xoy) + (xoy) = 0$ for all $x, y \in I$. In view of Lemma 2.1, we get the required result.

Theorem 2.3 *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d satisfying $F(xy) = xF(y) + d(x)y$ for all $x, y \in R$. Further, if $[F(x), d(y)] = [x, y]$ for all $x, y \in I$ or if $[F(x), d(y)] + [x, y] = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: We have

$$[F(x), d(y)] = [x, y] \text{ for all } x, y \in I.$$

Replacing x by xy and using the hypothesis, we arrive at

$$F(x)[y, d(y)] + [x, d(y)]d(y) = 0 \text{ for all } x, y \in I. \quad (2.9)$$

Now, further replacing x by yx in (2.9) and using (2.9) we find that $d(y)x[y, d(y)] + [y, d(y)]xd(y) = 0$ and hence by Lemma 2.3, we get $d(y)x[y, d(y)] = 0$ for all $x, y \in I$ and hence $[y, d(y)]x[y, d(y)] = 0$ that is, $[y, d(y)]I[y, d(y)] = \{0\}$ for all $y \in I$. Thus using similar method as used in the proof of Theorem 2.1, we find that $[y, d(y)] = 0$ for all $y \in I$. Now by the Lemma 2.2, we get the required result. □

In the event $[F(x), d(y)] + [x, y] = 0$ for all $x, y \in I$, it is equally easy to establish that $d(y)x[y, d(y)] + [y, d(y)]xd(y) = 0$ for all $x, y \in I$, therefore our proof is complete.

Theorem 2.4 *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d satisfying $F(xy) = xF(y) + d(x)y$ for all $x, y \in R$. Further, if $F([x, y]) = [d(x), F(y)]$ for all $x, y \in I$ or if $F([x, y]) + [d(x), F(y)] = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: We have
$$F([x,y]) = [d(x), F(y)] \text{ for all } x, y \in I.$$
Replacing y by xy and using the above relation, we find that
$$d(x)[x,y] = [d(x),x]F(y) + d(x)[d(x),y] \text{ for all } x, y \in I. \tag{2.10}$$
Again replace y by yx in (2.10) and use (2.10), to get $[d(x),x]yd(x) + d(x)y[d(x),x] = 0$ and hence by Lemma 2.3, we obtain that $[d(x),x]yd(x) = 0$ for all $x,y \in I$. This yields that $[d(x),x]I[d(x),x] = \{0\}$ for all $x \in I$. Thus using similar approach as used in the proof of Theorem 2.1, we find that $[d(x),x] = 0$ for all $x \in I$. Thus by Lemma 2.2, we get the required result. □

Also if we have $F([x,y]) + [d(x),F(y)] = 0$ for all $x,y \in I$, then it is easy to find that $[d(x),x]yd(x) + d(x)y[d(x),x] = 0$ for all $x,y \in I$, therefore our proof is complete.

Theorem 2.5 *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g respectively. If $[F(x),y] = [x,G(y)]$ for all $x,y \in I$ or if $[F(x),y] + [x,G(y)] = 0$ for all $x,y \in I$, then R contains a nonzero central ideal.*

Proof: We have
$$[F(x),y] = [x,G(y)] \text{ for all } x, y \in I.$$
Replacing y by yx in the above expression, we obtain
$$y[F(x),x] = [x,y]g(x) + y[x,g(x)] \text{ for all } x, y \in I. \tag{2.11}$$
Again replace y by zy in (2.11) and use (2.11), to get $[x,z]yg(x) = 0$ for all $x,y,z \in I$. Replacing z by $g(x)z$, we get $[x,g(x)]zR[x,g(x)]z = \{0\}$ for all $x,z \in I$ and hence by semiprimeness of R we find that $[x,g(x)]z = 0$ for all $x,z \in I$. This can be written as $[x,g(x)]R[x,g(x)] = \{0\}$ for all $x \in I$ and hence by semiprimeness of R we obtain $[x,g(x)] = 0$ for all $x \in I$. Thus by Lemma 2.2, we get the required result. □

Further, if $[F(x),y] + [x,G(y)] = 0$ for all $x,y \in I$, then using the same techniques as used above with necessary variations we get the required result.

Proceeding on the same lines with necessary variations and taking $G = F$ or $G = -F$ in Theorem 2.5, one can prove the following theorem.

Theorem 2.6 *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $[F(x),y] = [x,F(y)]$ for all $x,y \in I$ or if $[F(x),y] + [x,F(y)] = 0$ for all $x,y \in I$. Then R contains a nonzero central ideal.*

Theorem 2.7 *Let R be a semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $d(x)F(y) = xy$ for all $x, y \in I$ or if $d(x)F(y) + xy = 0$ for all $x, y \in I$. Then R contains a nonzero central ideal.*

Proof: By the given hypothesis, we have

$$d(x)F(y) = xy \text{ for all } x, y \in I.$$

Replacing y by yx in the last relation and using it, we get

$$d(x)yd(x) = 0 \text{ for all } x, y \in I. \tag{2.12}$$

This implies that $d(x)xyd(x) = 0$ for all $x, y \in I$. Further (2.12) also yields that $xd(x)yd(x) = 0$ for all $x, y \in I$. Combining this with the later relation we have $[d(x), x]yd(x) = 0$. Using similar procedure again it can be easily seen that $[d(x), x]y[d(x), x] = 0$ for all $x, y \in I$. This implies that $I[d(x), x]RI[d(x), x] = \{0\}$ and the semiprimeness of R yields that $I[d(x), x] = 0$ for all $x \in I$. Hence for any $r \in R$, $x \in I$, we find that $[r, x][d(x), x] = 0$. Again replacing r by $d(x)r$, we get $[d(x), x]R[d(x), x] = \{0\}$ for all $x \in I$. This implies that $[d(x), x] = 0$ for all $x \in I$ and hence by Lemma 2.2, we get the required result. □

On the otherhand, if $d(x)F(y) + xy = 0$ for all $x, y \in I$, then using the same techniques as used above with necessary variations we get the required result.

Theorem 2.8 *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $F([x, y]) = [F(x), y] + [d(y), x]$ for all $x, y \in I$. Then R contains a nonzero central ideal.*

Proof: For all $x, y \in I$, we have

$$F([x, y]) = [F(x), y] + [d(y), x]. \tag{2.13}$$

Replacing y by yx in (2.13) and using (2.13), we find that

$$2[x, y]d(x) = y[F(x), x] + y[d(x), x] \text{ for all } x, y \in I. \tag{2.14}$$

Now replace y by yz in (2.14) and use (2.14), to get $2[x, y]zd(x) = 0$ for all $x, y, z \in I$. Since R is 2-torsion free , we get $[x, y]zd(x) = 0$ for all $x, y, z \in I$. Replacing y by $d(x)y$, we get

$$[x, d(x)]yzd(x) = 0 \text{ for all } x, y, z \in I. \tag{2.15}$$

Replacing z by zx in (2.15), we get

$$[x, d(x)]yzxd(x) = 0 \text{ for all } x, y, z \in I. \tag{2.16}$$

Right multiplication of (2.15) by x gives

$$[x, d(x)]yzd(x)x = 0 \quad \text{for all } x, y, z \in I. \tag{2.17}$$

Combining (2.16) and (2.17), we get $[x, d(x)]yz[x, d(x)] = 0$ for all $x, y, z \in I$. Replacing z by zr where $r \in R$, the last expression can be written as $[x, d(x)]yzR[x, d(x)]yz = \{0\}$ for all $x, y, z \in I$. By semiprimeness of R we get $[x, d(x)]yz = 0$ for all $x, y, z \in I$. This implies that $[x, d(x)]yR[x, d(x)]y = \{0\}$ for all $x, y \in I$. Again by the semiprimeness of R, we find that $[x, d(x)]y = 0$ for all $x, y \in I$ and hence using similar method as used in the proof of Theorem 2.1, we find that $[x, d(x)] = 0$ for all $x \in I$. Now by the Lemma 2.2, we get the required result.

□

Theorem 2.9 *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all $x, y \in I$. Then R contains a nonzero central ideal.*

Proof: We have

$$F(x \circ y) = F(x) \circ y - d(y) \circ x \quad \text{for all } x, y \in I. \tag{2.18}$$

Replacing y by yx in (2.18) and using (2.18), we find that

$$(x \circ y)d(x) = -y[F(x), x] - y(d(x) \circ x) + [y, x]d(x) \quad \text{for all } x, y \in I. \tag{2.19}$$

Replace y by zy in (2.19) and use (2.19), to get $2[x, z]yd(x) = 0$ for all $x, y, z \in I$. Since R is 2-torsion free , we get $[x, z]yd(x) = 0$ for all $x, y, z \in I$. Replacing z by $d(x)z$, we get $[x, d(x)]zyd(x) = 0$ for all $x, y, z \in I$. Now by the same techniques as used in the Theorem 2.8, we get the required result. □

The following example shows that the condition that the ring R to be semiprime is not superfluous:

Example 2.1 *Consider T be any ring and let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \,\middle|\, x, y \in T \right\}$ and let $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \,\middle|\, y \in T \right\}$ be an ideal of R. Define $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that F is a generalized derivation with associated derivation d such that $d(x) = e_{11}x - xe_{11}$. It is straight forward to see that F satisfies the properties: $(i)[F(x), x] = 0$, $F(x) \circ x = 0$, (ii) $[F(x), d(y)] = [x, y]$, $(iii) F([x, y]) = [d(x), F(y)]$, $(iv) d(x)F(y) = xy$, $(v) F(x^2) = x^2$ (vi), $F([x, y]) = [F(x), y] + [d(y), x]$ and (vii) $F(x \circ y) = F(x) \circ y - d(y) \circ x$ for all $x, y \in I$. However, I is not central.*

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