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Improved numerical solution of Burger's equation

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abstract: In this paper we give the numerical solution of the Burger's equation using the variational iteration method (abbr. VIM) and we compare it with that of radial basis functions [6]. We remark an improvement of the numerical solution, next we compare the exact solution with the approximate solution by VIM in a given time interval.

Key Words: Burger's equation, variational iteration method, radial basis functions (abbr. RBFs).

Contents

1. Introduction

We consider the following Burger's equation:

$$u_t + uu_x - \nu u_{xx} = 0 \tag{1}$$

with the initial condition:

$$u(x,0) = \frac{\{\alpha + \beta + (\beta - \alpha)e^{\gamma}\}}{(1 + e^{\gamma})}, \tag{2}$$

where $\gamma = (\frac{\alpha}{\nu})(x - \eta)$ and α, β, η, ν are the parameters. The exact solution of the above problem is given in [13] by:

$$u(x,t) = \frac{[\alpha + \beta + (\beta - \alpha)\exp(\zeta)]}{\{1 + \exp(\zeta)\}}, \tag{3}$$

where $\zeta = (\frac{\alpha}{\nu})(x - \beta t - \eta)$.
This model arises in many physical applications such as propagation of waves in shallow water, propagation of waves in elastic tube filled with a viscous fluid [6].

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Many researchers have proposed various kinds of numerical methods for solving Burger's equation, we cite for example Meshfree method which is called element-free characteristic Galerkin method [17], in general there are many methods that belong to one of the following categories: finite difference method [14,7,16,12], finite element method [15,4,1], boundary element method [3], spectral methods [5]. In 2009, S. Haq, SU M. Uddin and Islam have given in [6], a method that uses radial basis functions to approximate the solution of (1) with condition (2).

Recently the variational iteration method was introduced by Ji-Huan He, in following work [8,9,10,11], is an effective procedure for solving various nonlinear problems without the usual restrictive assumptions it's used by many numerical analysts. We propose here this method for solving numerically equation (1) with condition (2) and we compare the results with those given by the method of radial basis functions [6], we deduce that there is an important improvement of the approximate solution. Then we present a comparison between the approximate solution which uses the VIM and the exact solution of equation (1) with condition (2) in a given time interval. This paper is organized as follows. In Section 2, we give a brief introduction of the variational iteration method. In Section 3, we apply the variational iteration method to give an approximate solution of Burger's equation. In Section 4, in first, the results obtained are compared with those using radial basis functions, we show that these results are better. In second, we give the approximate solution of Burger's equation by VIM in a given time interval.

2. A brief introduction of the variational iteration method

We consider the following nonlinear equation:

$$Lu + Nu = g(x,t) \tag{4}$$

where L is linear operator, N is a nonlinear operator and g is a known analytical function.
According to variational iteration method, we can construct a correction functional as follow:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(Lu_n(\xi) + N\tilde{u}(\xi) - g(\xi))d\xi \tag{5}$$

where λ is general Lagrangian multiplier, the subscript n denotes the n^{th} order approximation, u_0 is an initial approximation which can be known according to the initial conditions or the boundary conditions and $\tilde{u}_n$ is considered as restricted variation i.e., $\delta\tilde{u}_n = 0$. λ can be identified optimally via the variational theory, find the exact solution u with $u(x,t) = \lim_{n\to+\infty} u_n(x,t)$. According to iterations of this sequence, we can determine approximations of the solution, this is the procedure we will use in the comparison with the method of radial basis functions.

3. Application of method VIM to Burger's equation

To solve the equation (1) with initial condition (2) by the VIM, the correction functional can be written as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(\frac{\partial u_n}{\partial \xi}(x,\xi) + \tilde{u}_n \frac{\partial \tilde{u}_n}{\partial x}(x,\xi) - \nu \frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,\xi))d\xi. \quad (6)$$

As we have $\delta \tilde{u}_n = 0$, we can deduce

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(\xi)(\frac{\partial \delta u_n}{\partial \xi}(x,\xi))d\xi = 0.$$

via integration by parts, we finds

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda \delta u_n|_{\xi=t} - \int_0^t \lambda'(\xi)\delta u_n(x,\xi)d\xi = 0.$$

To find optimal value of λ, we deduce $\lambda(\xi) = 0$ and $1 + \lambda(\xi)|_{\xi=t} = 0$. From which the lagrangian multiplier can be identified as $\lambda = -1$, so the following iteration formula is obtained

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (\frac{\partial u_n}{\partial \xi}(x,\xi) + u_n \frac{\partial u_n}{\partial x}(x,\xi) - \nu \frac{\partial^2 u_n}{\partial x^2}(x,\xi))d\xi, \quad n \in \mathbb{N}. \quad (7)$$

with the initial condition $u_0 = u(x,0)$.

4. Numerical results

For numerical computations we choose $\alpha = 0.4$, $\beta = 0.6$, $\eta = 0.125$, $\nu = 1$ and $t = 1$ in order to compare the error of our method with that given in [6].

4.1. COMPARISON OF RESULTS. The graphs of errors by using the method of radial basis functions are given in [6]: Gaussian (GA) see Figure (1), inverse quadric (IQ) see Figure (2) and Multiquadric (MQ) see Figure (3). We give the error by using VIM method after four iteration.

The error by VIM in $J = [-20,-10] \cup [10,20]$ is very small and in $I = [-10,10]$ is between 10^{-7} and 2.9×10^{-6} see Figure (4), however the errors of methods IQ and GA in the interval $I = [-10.10]$ is between 10^{-3} and 1.3×10^{-2} see Figures (1,2) and for MQ method the error in the interval $I = [-10.10]$ is between 0.4×10^{-5} and 2.5×10^{-5} see Figure (3). We observe that the method of VIM gives a good improvement of the error see Figure (4).

4.2. APPROXIMATE SOLUTION OF BURGER'S EQUATION BY VIM METHOD. With the same data as before, by taking t in the time interval $[0,4]$, see Figure (5).

We note that when the time is increasing in the interval $I = [0,4]$, the error becomes great and especially when x in $[-10,10]$ see Figure (5).

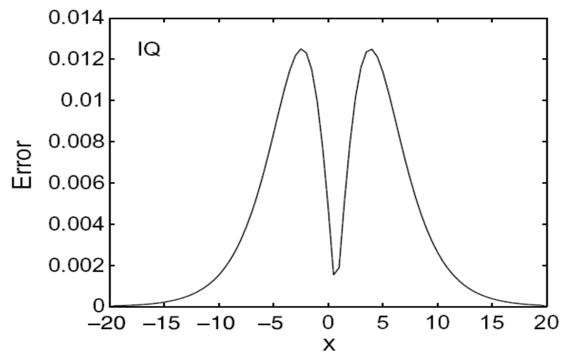

Figure 1: Graph of the error by using IQ method.

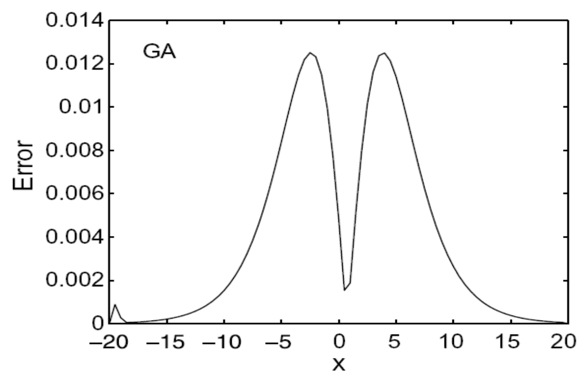

Figure 2: Graph of the error by using GA method.

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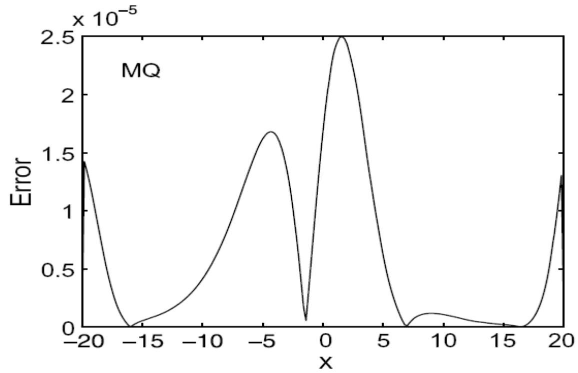

Figure 3: Graph of the error by using MQ method.

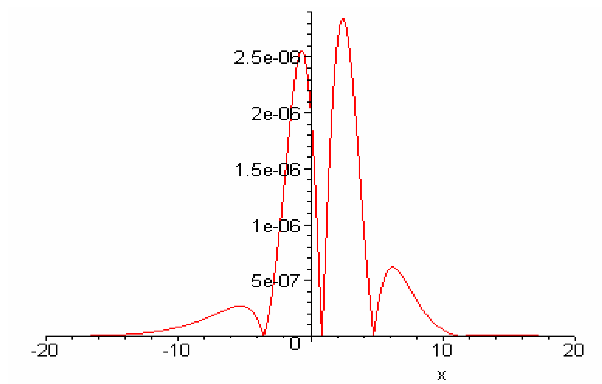

Figure 4: Graph of the error by using VIM method.

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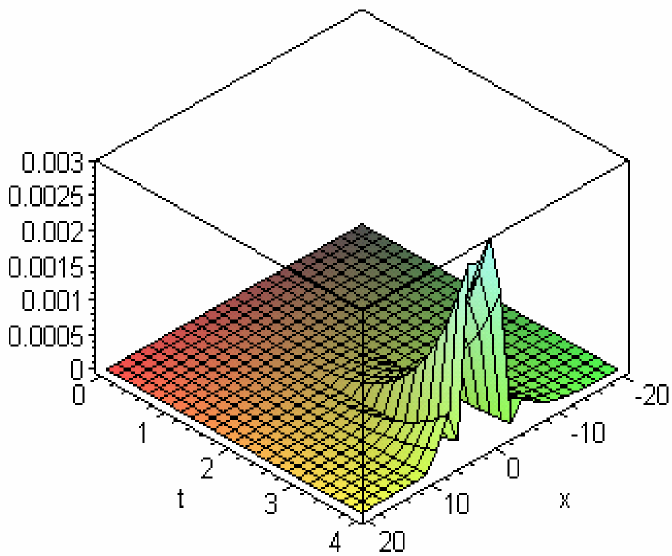

Figure 5: Graph of the error by using VIM method in interval of time [0,4].

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