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Pick's theorem in the classroom

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ABSTRACT: In this work we consider the Pick's theorem and some of its implications about area and perimeter of a plane region.

Key Words: Area, perimeter, teaching, Pick's theorem.

Contents

1. Introduction

The purpose of this work is to provide some considerations about the teaching of elementary mathematics, in particular, area and perimeter. In [4] we have the result of a research involving two groups of Chinese (72) and American (23) elementary mathematics teachers. They were asked if it was correct to say that there is some relation between area and perimeter of a polygon. There were many wrong answers to this simple question: 20 American teachers and 22 Chinese teachers did not know the right answer. That research tells us that we need show to our students that there are no relation between area and perimeter.

In [3] we find very interesting story: a teacher drew several polygons in the classroom, using juxtaposed copies of a pattern square, with same area of 16 units but different perimeters. These polygons were like figure 1. As for homework the teacher asked to calculate their areas. A student found out that all given areas in the homework were equal to $A = I + 7$, where I is the number of the interior intersection points. Surprised with this observation, the teacher had some questions. Is it true for all figures? What happens if the the boundary of figure is changed but the area is conserved?

2. The Pick's theorem

A figure P such as the one illustrated in Figure 1 is so-called PL-figure lattice. Let us denote $A(P)$ the area of P. Let $B(P) = \#\left(\partial P \cap \mathbb{Z}^2\right)$ be the number of lattice points in the boundary ∂P and $I(P) = \#\left(\breve{P} \cap \mathbb{Z}^2\right)$ be the number of lattice

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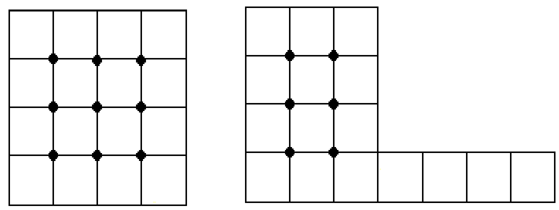

Figure 1: Two polygons: same area and different perimeter

points on the interior $\breve{P}$. Then $B(P)+I(P) = \#(P\cap\mathbb{Z}^2)$. There is a very important relation between $A(P), B(P)$ and $I(P)$ as we will see later .

The following formulae is known as Pick's theorem. See [1]. Georg Alexander Pick was born in 1859 in Vienna and died around 1943 in a Nazi concentration camp. His theorem was first published in 1899.

Theorem 2.1 (Pick) *Let P be a polygon in $\mathbb{R}^2$ using juxtaposed copies of a pattern square. Then*

$$A(P) = I(P) + \frac{B(P)}{2} - 1. \tag{1}$$

The following corollary answers the questions about the homework described in the Introduction.

Corollary 2.1A *The area of a polygon P with perimeter L satisfy the following relation*

$$A(P) - I(P) = \frac{L}{2} - 1.$$

Proof: Since the boundary ∂P is homeomorphic to a circle, then $B(P) = L$. Combining with the Pick's theorem, we have that

$$A(P) = I(P) + \frac{B(P)}{2} - 1 = I(P) + \frac{L}{2} - 1.$$

Thus we conclude the result. □

In particular, for the left hand side of Figure in 1 we have $\frac{L}{2} - 1 = 8 - 1 = 7$, so $A = I + 7$. For the right hand side of figure in 1 we have $\frac{L}{2} - 1 = 11 - 1 = 10$, so $A = I + 10 = 16$. In both cases the area is 16. This corollary answers the teacher.

3. Pick's formula generalizations

In general, a figure such as the illustrated in Figure 1 is usually called PL-figure lattice. More precisely, a PL-figure lattice is a compact subset P of $\mathbb{R}^2$ such that the boundary ∂P is a polygon whose vertices are in $\mathbb{Z}^2$, that is, integer coordinates.

Then the boundary ∂P is homeomorphic to a circle, a Jordan curve. A planar region under these conditions is called a lattice polygonal region. In Figure 2 is illustrated some PL-figure lattice.

For a PL-figure lattice P the formula (1) is known as Pick's theorem. See [1].

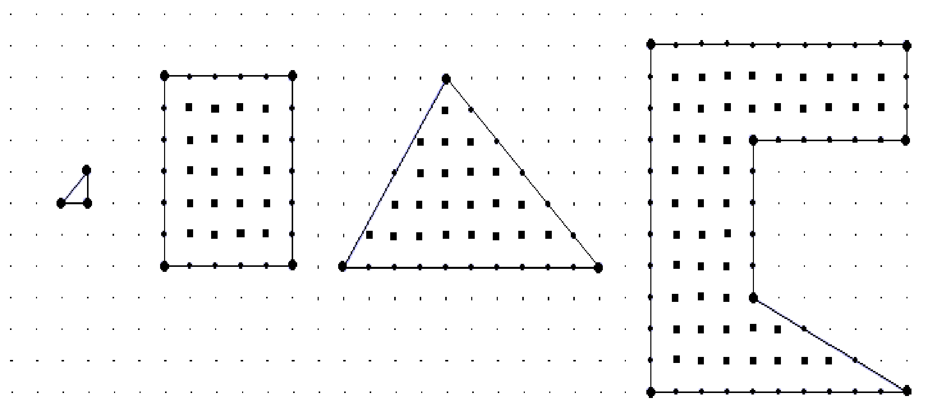

Figure 2: Integer coordinates

There are several results that are connected to Pick' theorem, see [2], [3], [5] and [6] and references therein. For instance, the Pick's theorem is a consequence of Homma's PL Gauss-Bonet theorem.

If we can consider more general polygons, which have vertices in $\mathbb{Z}^2$ and not only using juxtaposed copies of a pattern square, the Pick's formula in (1) does not hold. But there are generalizations of Pick's theorem given in [3]. We present two of them.

Theorem 3.1 *Let P be a polygonal lattice region in $\mathbb{R}^2$ such that closure$(\check{P}) = P$. If the boundary ∂P is homeomorphic to the disjoint union of several circles, then*

$$A(P) = I(P) + \frac{B(P)}{2} - \chi(P),$$

where $\chi(P)$ denotes the Euler number of figure P.

We note that in Figure 2 we have $\chi = 1$ for all figures. However Theorem 3.1 does not hold for polygonal regions with holes. In Figure 3 we have $I = 4$, $B = 20$ and $\chi = 0$, then, we have $A = 14$. For these cases we introduce the Theorem 3.2. See [2] and [5].

Theorem 3.2 (Reeve, Rosenholtz, Varberg) *Let P be a polygonal lattice region in $\mathbb{R}^2$ such that closure$(\check{P}) = P$. If the boundary ∂P is homeomorphic to the disjoint union of several circles, then*

$$A(P) = I(P) + \frac{B(P)}{2} + \frac{\chi(\partial P)}{2} - \chi(P),$$

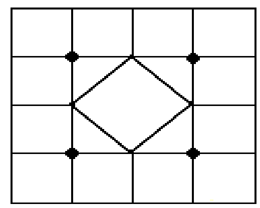

Figure 3: More general

where $\chi(X)$ denotes the Euler number of figure X.

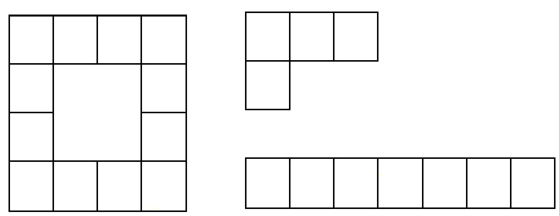

Figure 4: Polyominoes: union and hole

The proofs of these results can be found in the references. See [2], [3], [5] and [6] and references therein.

4. Suggested approaches in the classroom

In the early grades of elementary school can be proposed simple activities such as drawing some polygons with vertices integers in graph paper. First we can fix area and change the perimeter and ask the area and perimeter. After, we can fix the perimeter and change the area and ask the area and perimeter. We must conclude this activity with a remark that in general does not exist relation between area and perimeter.

In another activity, we can draw some polygons with vertices integers in a graph paper and then ask the number of border points B and the number of points inside a polygon I and fill a table with I, $\frac{B}{2}$, $I + \frac{B}{2} - 1$ and A. We must conclude this activity with the statement of the Pick's theorem in its simple case.

There is a Java procedure in *www.dma.uem.br/kit/textos/pick/pick-doc.html* due to W. T. Zenon and D. Andrade that let us draw polygons and apply the Pick's theorem in the simple case online. This software can be an important motivation.

We can use or introduce an intuitive idea of limit when we approximate the area of a region by a polygon. See the Figure 5. If we superimpose a mesh to the region and approximate it by a polygon we have a good approximation to this area. The approximation is better if we refine the mesh.

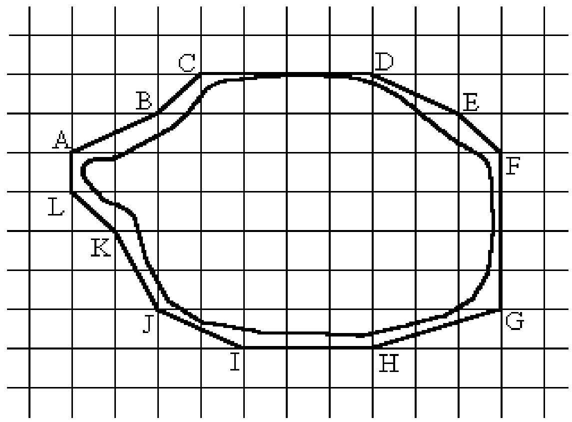

Figure 5: approximate area

We also use satellite photos to estimate areas on Earth. In this case, we will necessarily work with the notion of scale.

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