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An Asymmetric Steklov Problem With Weights: the singular case

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ABSTRACT: We prove the existence of a first nonprincipal eigenvalue for an asymmetric Steklov problem with weights. We are interested in the singular case (in where one of the weights has meanvalue zero), this case requires some special attention in connection with the Palais Smale (PS) conditions and with the mountain pass geometry.

Key Words: Steklov problem; p-Laplacian; Asymmetry; weights; Nonprincipal eigenvalue; Cerami (PS) condition.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a Lipschitz continuous boundary. Let $1 < p < \infty$ and let $\frac{N-1}{p-1} < q < \infty$ if $p < N$ and $q \geq 1$ if $p \geq N$. $m, n \in L^q(\partial\Omega)$ with $m^+ \neq 0$ and $n^+ \neq 0$. The asymmetric Steklov problem is defined by

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda[m(u^+)^{p-1} - n(u^-)^{p-1}] & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}^+$ is the eigenvalue, $u \in W^{1,p}(\Omega)$ is an associated eigenvalue and ν is the unit exterior normal. The solutions of (1) or of related equations are always understood in the weak sense, i.e., $u \in W^{1,p}(\Omega)$ with

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \lambda \int_{\partial\Omega} [m(u^+)^{p-1} - n(u^-)^{p-1}]\varphi d\sigma \quad \forall \varphi \in W^{1,p}(\Omega), \tag{2}$$

where $d\sigma$ is the $N-1$ dimensional Hausdorff measure. In a previous work (see [1]), we proved the existence of a first nonprincipal eigenvalue for (1) are $\int_{\partial\Omega} m d\sigma \neq 0$ and $\int_{\partial\Omega} n d\sigma \neq 0$ by applying a version of the mountain pass theorem to the functional $f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$ restricted to the manifold

$$M_{m,n} := \{u \in W^{1,p}(\Omega); \frac{1}{p}\int_{\partial\Omega} [m(u^+)^{p-1} + n(u^-)^{p-1}] d\sigma = 1\},$$

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in this case (PS) condition is satisfied and the geometry of the mountain pass was derived from observation that φ_m and φ_n where strict local minima (φ_m denotes the positive first eigenvalue of (1) with $m = n$). Our purpose in this work is to prove the existence of a first nonprincipal eigenvalue for (1) where $\int_{\partial\Omega} m d\sigma = 0$ or $\int_{\partial\Omega} n d\sigma = 0$. In this case the Palais Smale condition is not satisfied any more at level 0 and at least one of the two naturals candidates for local minimum fails to belong to the manifold $M_{m,n}$. To by pass this difficulty we apply a version of the mountain pass theorem for a local C^1 functional restricted to a C^1 manifold and which satisfies the Palais-Smale condition of Cerami (PSC) at certain levels(see [2]).

2. Preliminaries

Our main purpose in this preliminaries section, is to collect some results relative to the following eigenvalue problem

$$\triangle_p u = 0 \text{ in } \Omega, |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m|u|^{p-2}u \text{ on } \partial\Omega. \tag{3}$$

Clearly 0 is a principal eigenvalue of (3) with the constants as eigenfunctions. The search for another principal eigenvalue involves the following quantity

$$\lambda_1^*(m) = \inf\{\frac{1}{p}\int_\Omega |\nabla u|^p; u \in W^{1,p}(\Omega) \text{ and } \frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma = 1\}, \tag{4}$$

we have $\lambda_1^*(m) < \infty$ since $m^+ \neq 0$ in Ω.

Proposition 2.1 *1. If $\int_\Omega m d\sigma < 0$. Then $\lambda_1^*(m) > 0$ is the first positive Steklov eigenvalue. Moreover $\lambda_1^*(m)$ is simple and isolated and it is the only nonzero Steklov eigenvalue associated to an eigenfunction of definite sign .*

2. If $\int_\Omega m d\sigma > 0$. Then $\lambda_1^*(m) = 0$ and 0 is the unique nonnegative principal eigenvalue.

3. If $\int_\Omega m d\sigma = 0$. Then $\lambda_1^*(m) = 0$ and 0 is the unique principal eigenvalue.

Proposition 2.1 is proved in [4] (see also [1]). In case 1 or 2 of Proposition 2.1, the infimum is achieved at $\varphi_m \in M_{m,n}$ the positive eigenfunction associated to $\lambda_1^*(m)$ with $\frac{1}{p}\int_{\partial\Omega} m\varphi_m^p = 1$. In the case 3 the fact that $\lambda_1^*(m) = 0$ is easily verified by considering the sequence

$$v_k = \frac{(1+\psi/k)^{\frac{1}{p}}}{[\frac{1}{p}\int_{\partial\Omega} m(1+\psi/k)]^{\frac{1}{p}}}, \tag{5}$$

where ψ is any fixed smooth function with $\psi \geq 0$ and $\int_{\partial\Omega} m\psi > 0$. Note that in case 3 of Proposition 2.1, the infimum in (4) is not achieved (since no constant satisfies the constraint in that case). To get a first nonprincipal eigenvalue for an asymmetric Steklov problem with weights, we will use a version of the mountain pass theorem on C^1 manifold, which we now recall. Let E be a real Banach space and let $M := \{u \in E; g(u) = 1\}$, where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$ and consider the restriction $\tilde{f}$ of f to M.

Proposition 2.2 *([3]) Assume $\tilde{f}$ bounded from below and let*

$$c = \inf\{f(u); u \in M\}.$$

Then $\tilde{f}$ satisfies $(PSC)_c$ if and only if $\tilde{f}$ satisfies $(PS)_c$.

Remark 2.1 Going back to case 3 of Proposition (2.1), one can see that the functional $f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$ restricted to the manifold $M_{m,n}$ does not satisfy the $(PS)_0$. Indeed the sequence v_k from (5) provides an unbounded (PS) sequence. That the $(PSC)_0$ condition does not hold neither will follow from Proposition 2.2.

Proposition 2.3 *([3]) Let K be a compact metric space, $K_0 \subset K$ and $h_0 \in C(K_0, M)$. Consider the family of extensions of h_0: $\boldsymbol{H} := \{h \in C(K, M) : h/K_0 = h_0\}$. Assume $\boldsymbol{H}$ nonempty as well as the following condition $\max\limits_{t\in K_0} f(h_0(t)) < \max\limits_{t\in K} f(h(t))$ for any $h \in \boldsymbol{H}$. Define*

$$c := \inf_{h\in \boldsymbol{H}} \max_{t\in K} f(h(t)). \tag{6}$$

Assume that $\tilde{f}$ satisfies $(PSC)_c$ for c given in (6). Then c is a critical value of $\tilde{f}$.

Typically, as in the application $K = [0, 1]$, $K_0 = \{0, 1\}$, $f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$, $E = W^{1,p}(\Omega)$ and $g(u) = \frac{1}{p}\int_{\partial\Omega}[m(u^+)^{p-1} + n(u^-)^{p-1}]d\sigma$.

3. A first nontrivial eigenvalue

The assumptions on m, n in this section are m, $n \in L^q(\partial\Omega)$ with $\int_{\partial\Omega} m = 0$ or $\int_{\partial\Omega} n = 0$ and $m^+ \neq 0$, $n^+ \neq 0$. We look for nonnegative eigenvalues λ of (1). Clearly the only nonnegative principal eigenvalues of (1) are 0, $\lambda^*(m)$ and $\lambda^*(n)$. Moreover multiplying by u^+ or u^- one easily sees that if (1) with $\lambda \geq 0$ has a solution which changes sign then $\lambda > \max(\lambda^*(m), \lambda^*(n))$. Proving the existence of such a solution which changes sign and which in addition corresponds to a minimum value of λ is our purpose in this section. We will use a variational approach and consider the functional $f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$ on $E = W^{1,p}(\Omega)$, the manifold $M_{m,n}$ defined in introduction and the restriction $\tilde{f}$ of f to $M_{m,n}$. To state our main result let us introduce the following family of paths in $M_{m,n}$
$\Gamma = \{\gamma \in C([0, 1], M_{m,n}) : \gamma(0) \leq 0 \text{ and } \gamma(1) \geq 0\}$, which is nonempty (see [1]), and the finite minimax value

$$c(m, n) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[0,1]} \tilde{f}(u). \tag{7}$$

Theorem 3.1 *Assume $\int_{\partial\Omega} m = 0$ or $\int_{\partial\Omega} n = 0$. Then $c(m, n)$ is an eigenvalue of (1) which satisfies $\max\{\lambda_1^*(m), \lambda_1^*(n)\} < c(m, n)$. Moreover there is no eigenvalue of (1) between $\max\{\lambda_1^*(m), \lambda_1^*(n)\}$ and $c(m, n)$.*

The rest of this section is devoted to the proof of Theorem 3.1.

Proposition 3.1 *$\tilde{f}$ satisfies $(PSC)_c$ for all $c > 0$.*

Proof. Let $u_k \in M_{m,n}$ be a $(PSC)_c$ sequence for $\tilde{f}$, with $c > 0$. So $\int_\Omega |\nabla u_k|^p dx \to c$ and

$$|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla \xi dx| \leqslant \frac{\epsilon_k}{1+\|u_k\|}\|\xi\| \quad \forall \xi \in T_{u_k}M_{m,n} \tag{8}$$

where $\epsilon_k \to 0$. We will show that u_k remains bounded and concludes that u_k admits a convergent subsequence. Let us assume by contradiction that for a subsequence, $\|u_k\| \to \infty$. Write $v_k = \frac{u_k}{\|u_k\|}$. For a further subsequence, $v_k \to v_0$ weakly in $W^{1,p}(\Omega)$. Since $\int_\Omega |\nabla u_k|^p dx$ remains bounded, one has $\int_\Omega |\nabla v_k|^p dx \to 0$ and it follows easily that $v_0 \equiv cst \neq 0$ and that $v_k \to v_0$ strongly in $W^{1,p}(\Omega)$. On the other hand, taking $\xi = a_k(\omega) := \omega - [\int_{\partial\Omega}(m(u_k^+)^{p-1} - n(u_k^-)^{p-1})\omega]u_k$ in (8), where $\omega \in W^{1,p}(\Omega)$ and dividing by $\|u_k\|^{p-1}$, one gets

$$\begin{aligned}&\left|\int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla\omega - [\int_{\partial\Omega}(m(v_k^+)^{p-1} - n(v_k^-)^{p-1})\omega]\int_\Omega |u_k|^p\right| \\ &\leqslant \epsilon_k \frac{\|u_k\|}{1+\|u_k\|}\left\|\frac{\omega}{\|u_k\|^p} - [\int_{\partial\Omega}(m(v_k^+)^{p-1} - n(v_k^-)^{p-1})\omega]v_k\right\|.\end{aligned}$$

By passing to the limit, we implies that v_0 is a solution of

$$-\Delta_p v_0 = 0 \text{ in } \Omega, \quad |\nabla v_0|^{p-2}\frac{\partial v_0}{\partial\nu} = c[m(v_0^+)^{p-1} - n(v_0^-)^{p-1}] \text{ on } \partial\Omega, \tag{9}$$

where c is the level appearing in the $(PSC)_c$ sequence. Since $v_0 \equiv cst$, the right-hand side of (9) is $\equiv 0$, and since $c > 0$, one gets $m(v_0^+)^{p-1} - n(v_0^-)^{p-1} \equiv 0$. This relation with a nonzero constant v_0 implies $m \equiv 0$ or $n \equiv 0$, which contradicts $m^+ \not\equiv 0$ and $n^+ \not\equiv 0$. Thus u_k remains bounded, for a subsequence, $u_k \to u_0$ weakly in $W^{1,p}(\Omega)$. Taking $\xi = a_k(\omega)$ in (8), one deduces

$$\begin{aligned}&\left|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla\omega - [\int_{\partial\Omega}(m(u_k^+)^{p-1} - n(u_k^-)^{p-1})\omega]\int_\Omega |u_k|^p\right| \\ &\leqslant \epsilon_k \frac{\|a_k(\omega)\|}{1+\|u_k\|} \leqslant D\epsilon_k \frac{\|u_k\|^p + 1}{\|u_k\|+1}\|\omega\|\end{aligned}$$

for some constant D; taking now $\omega = u_k - u_0$ in the above, one obtains $\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k - u_0) \to 0$. Since $\int_\Omega |u_k|^{p-2}u_k(u_k - u_0) \to 0$, it then follows from the (S^+) property that $u_k \to u_0$ strongly in $W^{1,p}(\Omega)$, which yields the conclusion. □

We now turn to the geometry of $\tilde{f}$. The situation here is again simpler in the non singular case (see Preliminaries). We start by giving an important proposition, which is proved in [1].

Proposition 3.2 *If $\int_{\partial\Omega} m d\sigma \neq 0$, then $\varphi_m \in M_{m,n}$ is a strict local minimum of $\tilde{f}$, with in addition for some $\epsilon_0 > 0$ and all $0 < \epsilon < \epsilon_0$,*

$$\tilde{f}(\varphi_m) = \lambda_1^*(m) < \inf\{\tilde{f}(u); u \in M_{m,n} \cap \partial B(\varphi_m, \epsilon)\}, \tag{10}$$

where $B(\varphi_m, \epsilon)$ denotes the ball in $W^{1,p}(\Omega)$ of center φ_m and radius ϵ. Similar conclusion for $-\varphi_n$ if $\int_{\partial\Omega} n d\sigma \neq 0$.

In the singular case, one at least of the two local minima provided by Proposition 3.2 is missing. The search for suitable endpoints of path which allow the application of a mountain pass argument will be based on the following lemmas (see in particular Lemma 3.4).

Lemma 3.1 *Assume $\int_{\partial\Omega} m d\sigma = 0$ or $\int_{\partial\Omega} n d\sigma = 0$. Then $\max\{\lambda^*(m), \lambda^*(n)\} < c(m,n)$.*

Proof. The inequality $\leqslant$ easily follows from the definition of $\lambda^*(m)$ and $\lambda^*(n)$. Indeed for any $\gamma \in \Gamma$, $\gamma(1) \in M_{m,n}$, is nonnegative and so satisfies the constraint in the definition of $\lambda^*(m)$. Consequently $c(m,n) \geqslant \lambda^*(m)$, and a similar argument applies to $\lambda^*(n)$. To prove the strict inequality assume by contradiction that for instance $\lambda^*(m) = c(m,n)$. So there exists a sequence $\gamma_k \in \Gamma$ such that

$$\max_{t\in[0,1]} \tilde{f}(\gamma_k(t)) \to \lambda^*(m). \tag{11}$$

Put $u_k := \gamma_k(1)$. Since $u_k \geqslant 0$, one has

$$\lambda^*(m) \leqslant \int_\Omega |\nabla u_k|^p \leqslant \max_{t\in[0,1]} \tilde{f}(\gamma_k(t)) \to \lambda^*(m), \tag{12}$$

and consequently $\int_\Omega |\nabla u_k|^p \to \lambda^*(m)$. Let us now distinguish two cases: First case, $\|u_k\|$ remains bounded, for a subsequence, $u_k \to u_0$ weakly in $W^{1,p}(\Omega)$. Since $u_k \in M_{m,n}$ and $u_k \geqslant 0$, one has $\int_{\partial\Omega} m|u_0|^p = 1$, and so $\lambda^*(m) \leqslant \int_\Omega |\nabla u_0|^p \leqslant \liminf \int_\Omega |\nabla u_k|^p = \lambda^*(m)$, which implies that $\int_\Omega |\nabla u_0|^p = \lambda^*(m)$. Consequently $u_k \to u_0$ strongly in $W^{1,p}(\Omega)$.
If $\int_{\partial\Omega} m d\sigma = 0$, then $\lambda^*(m) = 0$ and so $u_0 \equiv cst$, which leads to a contradiction with $\int_{\partial\Omega} m|u_0|^p = 1$. So $\int_{\partial\Omega} m d\sigma \neq 0$ and we conclude that $u_0 = \varphi_m$. Let us now choose $\epsilon > 0$ such that $\tilde{f}(\varphi_m) = \lambda^*(m) < \inf\{\tilde{f}(u) : u \in M_{m,n} \cap \partial B(\varphi_m, \epsilon)\}$, and $B(\varphi_m, \epsilon)$ does not contain any function v with $v \leqslant 0$, which clearly possible. For k sufficiently large $u_k := \gamma_k(1) \in B(\varphi_m, \epsilon)$, while $\gamma_k(0) \notin B(\varphi_m, \epsilon)$ since $\gamma_k(0) \leqslant 0$. It follows that the path γ_k intersects $\partial B(\varphi_m, \epsilon)$ and consequently $\max_{t\in[0,1]} \tilde{f}(\gamma_k(t)) \geqslant \inf\{\tilde{f}(u) : u \in M_{m,n} \cap \partial B(\varphi_m, \epsilon)\} > \lambda^*(m)$, this contradicts (10). Second case, $\|u_k\| \to \infty$, we put $v_k := \frac{u_k}{\|u_k\|}$. For a subsequence, $v_k \to v_0$ weakly in $W^{1,p}(\Omega)$. Since $\int_\Omega |\nabla u_k|^p$ remains bounded, we obtain $\int_\Omega |\nabla v_k|^p \to 0$ and so $v_0 \equiv cst \not\equiv 0$. Moreover $\int_{\partial\Omega} m|v_0|^p d\sigma = 0$ since $\int_{\partial\Omega} m|u_k|^p d\sigma = 1$. We have reached a contradiction if $\int_{\partial\Omega} m d\sigma \neq 0$. So let us assume from now on that $\int_{\partial\Omega} m d\sigma = 0$. We first observe that for any $\gamma \in \Gamma$ there exists $t_0 = t_0(\gamma) \in [0,1]$ such that

$$\frac{1}{p}\int_{\partial\Omega} m(\gamma(t_0)^+)^p d\sigma = \frac{1}{p}\int_{\partial\Omega} n(\gamma(t_0)^-)^p d\sigma = \frac{1}{2}. \tag{13}$$

Consider now $w_k := \gamma_k(t_0(\gamma_k))$. We have now instead of (12)

$$0 \leqslant \frac{1}{p}\int_\Omega |\nabla w_k|^p \leqslant \max_{t\in[0,1]} \tilde{f}(\gamma_k(t)) \to \lambda^*(m) = 0. \tag{14}$$

We again distinguish two case: First case, $w_k \to w_0$ weakly in $W^{1,p}(\Omega)$. It follows from (14) that $w_0 \equiv cst$ and that $w_k \to w_0$ strongly in $W^{1,p}(\Omega)$. A contradiction then follows from $\frac{1}{p}\int_{\partial\Omega} m(w_0^+)^p d\sigma = \frac{1}{p}\int_{\partial\Omega} n(w_0^-)^p d\sigma = \frac{1}{2}$. Second case, $\|w_k\| \to \infty$, we put $z_k := \frac{w_k}{\|w_k\|}$. For a subsequence $z_k \to z_0$ weakly in $W^{1,p}(\Omega)$. It follows from (14) that $z_0 \equiv cst$ and that $z_k \to z_0$ strongly in $W^{1,p}(\Omega)$, consequently $\|z_0\| = 1$. If $z_0 > 0$ (a similar argument applies if $z_0 < 0$), then $|w_k < 0| = |z_k < 0| \to 0$; moreover w_k changes sign and by (13) $\frac{\int_{\partial\Omega} n^+|w_k^-|^p}{\int_\Omega |\nabla w_k^-|^p} \geqslant \frac{\frac{1}{2}}{\int_\Omega |\nabla w_k|^p} \to +\infty$. This yields a contradiction with the following lemma.

Lemma 3.2 *(see [1]) Let $v_k \in W^{1,p}(\Omega)$ with $v_k \geq 0$, $v_k \not\equiv 0$ and $|v_k > 0| \to 0$. Let n_k be bounded in $L^q(\partial\Omega)$. Then $\frac{\int_{\partial\Omega} n_k v_k^p d\sigma}{\int_\Omega |\nabla v_k|^p dx} \to 0$.*

Lemma 3.3 *For any $d > 0$, the set $\mathbf{O} := \{u \in M_{m,n}; u \geq 0$ and $\tilde{A}(u) < d\}$ is arcwise connected. Similar conclusion if $u \geq 0$ is replaced by $u \leq 0$.*

Proof. Since $\mathbf{O}$ is empty if $d \leq \lambda_1^*(m)$, we can assume from now on $d > \lambda_1^*(m)$. The case where $\int_{\partial\Omega} m d\sigma \neq 0$ is proved in [1]. Consider now the case where $\int_{\partial\Omega} m d\sigma = 0$. Let u_1, $u_2 \in \mathbf{O}$. One starts by decreasing a little bit the weight m into a weight $\hat{m} \in L^q(\partial\Omega)$ such that $\hat{m} \leqslant m$, $\int_{\partial\Omega} \hat{m} d\sigma < 0$, $\int_{\partial\Omega} \hat{m} u_1^p d\sigma > 0$, $\int_{\partial\Omega} \hat{m} u_2^p d\sigma > 0$ and $\frac{\int_\Omega |\nabla u_1|^p}{\int_{\partial\Omega} \hat{m} u_1^p d\sigma} < d$, $\frac{\int_\Omega |\nabla u_2|^p}{\int_{\partial\Omega} \hat{m} u_2^p d\sigma} < d$, which is clearly possible since $\lambda^*(m) < d$. Put $v_1 := \frac{u_1}{(\frac{1}{p}\int_{\partial\Omega} \hat{m} u_1^p d\sigma)^{\frac{1}{p}}}$ and $v_2 := \frac{u_2}{(\frac{1}{p}\int_{\partial\Omega} \hat{m} u_2^p d\sigma)^{\frac{1}{p}}}$. By the first case, there exists a path $\gamma \in M_{\hat{m},\hat{m}}$ which goes from v_1, v_2, is made of nonnegative functions and is such that $f(\gamma(t)) < d$ for all t. Consider now the path $\gamma_1(t) := \frac{\gamma(t)}{(\frac{1}{p}\int_{\partial\Omega} m|\gamma(t)|^p d\sigma)^{\frac{1}{p}}}$. By the choice of $\hat{m}$,

$$\frac{1}{p}\int_{\partial\Omega} m|\gamma(t)|^p d\sigma \geqslant \frac{1}{p}\int_{\partial\Omega} \hat{m}|\gamma(t)|^p d\sigma = 1, \tag{15}$$

and consequently γ_1 is a well defined path in $M_{m,n}$, which clearly goes from u_1 to u_2 and is made of nonnegative functions. Moreover, by (15), $f(\gamma_1(t)) = \frac{f(\gamma(t))}{\frac{1}{p}\int_{\partial\Omega} m|\gamma(t)|^p d\sigma} \leqslant f(\gamma(t)) < d$ for all t. This concludes the proof of Lemma 3.3 for $\mathbf{O}$ with $u \geqslant 0$. Similar argument in the case $u \leq 0$. □

Lemma 3.4 *Assume $\int_{\partial\Omega} m d\sigma = 0$ or $\int_{\partial\Omega} n d\sigma = 0$. Then there exist $u_1 \geq 0$ and $u_2 \leq 0$ in $M_{m,n}$ such that $\tilde{f}(u_1) < c(m,n)$ and $\tilde{f}(u_2) < c(m,n)$. Moreover, for any such choice of u_1, u_2, one has*

$$c(m,n) = \inf_{\gamma\in\tilde{\Gamma}} \max_{u\in\gamma([0,1])} \tilde{f}(u) \tag{16}$$

where $\bar{\Gamma} := \{\gamma \in C([0,1], M_{m,n}); \gamma(0) = u_2$ *and* $\gamma(1) = u_1\}$ *and* $c(m,n)$ *is defined by* (7).

Proof. If $\int_{\partial\Omega} m d\sigma \neq 0$, one takes $u_1 = \varphi_m$ and the inequality $\tilde{f}(u_1) < c(m,n)$ follows from Lemma 3.1. Similarly with $u_2 = -\varphi_n$ in case $\int_{\partial\Omega} n d\sigma \neq 0$. If now $\int_{\partial\Omega} m d\sigma = 0$, one takes $u_1 = v_k$ for k sufficiently large, where v_k is defined in (5). Indeed $\tilde{f}(v_k) \to 0$ and by Lemma 3.1, $0 < c(m,n)$, so that $\tilde{f}(v_k) < c(m,n)$ for k sufficiently large. Similar argument for the choice of u_2 in case $\int_{\partial\Omega} n d\sigma = 0$. To prove the equality (16), one uses the Lemma 3.3 and the same argument given by Lemma 3.9 in [1]. □

We are now in position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.1, one has $c(m,n) > \max\{\lambda_1^*(m), \lambda_1^*(n)\}$. To prove that $c(m,n)$ is an eigenvalue, we pick u_1, u_2 as in Lemma 3.4 and we will show that $\bar{c}$, the right-hand side of (16), is a critical value of $\tilde{f}$. Since $\int_{\partial\Omega} m d\sigma = 0$ or $\int_{\partial\Omega} n d\sigma = 0$, we know (see Proposition 3.1) that $\tilde{f}$ satisfies $(PSC)_c$ for all $c > 0$ and the mountain pass theorem (see Proposition 2.3) yields the conclusion. To show that there is no eigenvalue between $\max\{\lambda^*(m), \lambda^*(n)\}$ and $c(m,n)$, we follow the same proof from the nonsingular case (see [1]).□

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