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Polynomial and Analytic Boundary Feedback Stabilization of Square Plate

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ABSTRACT: We consider a boundary feedback stabilization problem of the plate equation in a square, in the case where the geometric condition of Ammari-Tucsnak [6] is not satisfied. We prove a polynomial decay for regular initial data. Moreover, we prove an exponential stability result for some subspace of the energy space. Finally, we give a precise estimate on the analyticity of reachable functions where we have an exponential stability.

Key Words: Plate equation, polynomial stabilization, analytic stabilization, observability inequality, low and high frequency.

Contents

1. Introduction

Let $\Omega = (0,\pi)\times(0,\pi)\subset\mathbb{R}^2$. We denote by $\partial\Omega$ the boundary of Ω and we assume that $\partial\Omega = \Gamma_0\cup\Gamma_1$, where $\Gamma_0 = \{(0,y)/y\in(0,\pi)\}$ and $\Gamma_1 = \partial\Omega\setminus\overline{\Gamma}_0$.

We consider the plate equation as follows :

$$\begin{cases} \partial_t^2 u + \Delta^2 u = 0 & \Omega\times(0,+\infty), \\ \Delta u = -\frac{\partial}{\partial\nu}[G(\partial_t u)]1_{|\Gamma_0} & \partial\Omega\times(0,+\infty), \\ u = 0 & \partial\Omega\times(0,+\infty) \\ u(x,0) = u^0(x),\ \ \partial_t u(x,0) = u^1(x) & \Omega, \end{cases} \tag{1}$$

where the operator G is defined as $(-\Delta)^{-1} : H^{-1}(\Omega)\longrightarrow H_0^1(\Omega)$, ν is the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\Delta^2 : \mathcal{D}(\Delta^2)\longrightarrow H^{-1}(\Omega)$ be a self-adjoint, positive and boundedly invertible operator where

$$\mathcal{D}(\Delta^2) = \{u\in H^{-1}(\Omega)/\Delta^2 u\in H^{-1}(\Omega)\}.$$

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The controllability of the dynamical system modelling the vibrations of the plate with boundary control acting on the moment has been investigated in several works such as Ammari and Khenissi [4]- [10], Ammari and Vodev [9], Krabs, Leugering and Seidman [15], Leugering [20], Lebeau [17], [14] and in [23]. In [16] the exact controllability of the same system has been established under the assumption that Ω is a square and under much weaker assumption on the controlled part of the boundary (Γ_1 is only supposed to contain non-empty vertical and horizontal subsects). The geometric optics condition introduced by Bardos, Lebeau and Rauch in [12] for the wave equation is thus not necessary in this case. In fact, recently, Ammari and Tucsnak (see [6]) have proved that the system is exponentially stable if and only if the controlled part of the boundary contains a vertical and horizontal part of non-zero length.

In this work, we study the polynomial stability for regular initial data and we study the exponential stability for some analytic initial data of a square Euler-Bernoulli plate with feedback. We use the methodology introduced in [5] (see also [13] for the bounded case), where the exponential stability for this problem is reduced to an observability inequality proved by [23] :

$$\int_0^T \left\| \frac{\partial[G(\partial_t \phi)]}{\partial \nu} \right\|^2_{L^2(\Gamma_0)} dt \geq C \|\underline{u}\|^2_{H_0^1(\Omega)\times H^{-1}(\Omega)}, \quad \forall\, (u^0, u^1) \in H_0^1(\Omega) \times H^{-1}(\Omega), \tag{2}$$

where ϕ is the solution of the following undamped system associated to (1) :

$$\begin{cases} \partial_t^2 \phi + \Delta^2 \phi = 0 \;\; \Omega \times (0, +\infty), \\ \phi = 0 \;\; \partial\Omega \times (0, +\infty) \\ \phi(x, 0) = u^0(x), \;\; \partial_t \phi(x, 0) = u^1(x) \;\; \Omega. \end{cases} \tag{3}$$

The paper is organized as follows. The statements of the main results are given in the following section. Section 3 is devoted to the observability inequality of high and low frequency. In Section 4, we give some background on a class of dynamical systems. Finally, Section 5 contains the proof of main results.

2. Main results

The system (1) is well-posed for initial condition satisfying $(u^0, u^1) \in E = H_0^1(\Omega) \times H^{-1}(\Omega)$, i.e there exists a unique solution (see [6])

$$u \in C\big((0, +\infty), H_0^1(\Omega)\big) \cap C^1\big((0, +\infty), H^{-1}(\Omega)\big).$$

The energy $E(t)$ of system (1) is given by the following expression :

$$E(t) = \frac{1}{2}\left(\|u(t)\|^2_{H_0^1(\Omega)} + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} \right).$$

The solution of (1) satisfies the following energy estimate:

$$E(t) - E(0) = -\int_0^t \int_{\Gamma_0} \left| \frac{\partial[G(\partial_t u)]}{\partial \nu} \right|^2 d\Gamma_0 ds, \; \forall\, t \geq 0. \tag{4}$$

Let $n_0 \in \mathbb{N}^*$ fixed, we denote by E_{n_0} the following space :

$$E_{n_0} = \left\{ \sum_{(n,k)\in(\mathbb{N}^*)^2} a_{n,k}\varphi_{n,k} \in E,\, (a_{n,k}) \in l^2,\, a_{n,k} = 0,\, \forall\, k \neq n_0 \right\};$$

$$E^1 = \left\{ \sum_{(n,k)\in(\mathbb{N}^*)^2} a_{n,k}\varphi_{n,k} \in E,\, (a_{n,k}) \in l^2,\, a_{n,k} = 0,\, \forall\, k > n \right\};$$

$$E^2 = \left\{ \sum_{(n,k)\in(\mathbb{N}^*)^2} a_{n,k}\varphi_{n,k} \in E,\, (a_{n,k}) \in l^2,\, a_{n,k} = 0,\, \forall\, k \leq n \right\};$$

and by $E_n^i = E^i \cap E_n, \quad i = 1, 2$. Where

$$\varphi_{n,k} = \frac{2}{\pi}\sqrt{\lambda_{n,k}}\left(\sin ny \sin kx,\, i\lambda_{n,k} \sin ny \sin kx\right), \qquad \forall\, n, k \in \mathbb{N}^*$$

be the eigenfunctions sequence of Δ^2 normalized in $H_0^1(\Omega) \times H^{-1}(\Omega)$ and

$$\lambda_{n,k} = n^2 + k^2, \qquad \forall\, n, k \in \mathbb{N}^*$$

is the eigenvalues sequence of Δ^2.

For all $(u^0, u^1) \in E$, there exists $(a_{n,k}) \in l^2$ such that

$$(u^0, u^1) = \sum_{n,k\geq 1} a_{n,k}\varphi_{n,k}.$$

Which implies, $E = \bigoplus_{n\in\mathbb{N}^*} E_n$. For all $\alpha \in \mathbb{R}_+^*$, we define the following spaces (for more details, see [18]):

$$X_{0,\alpha} := \left\{ \sum_{(n,k)\in\mathbb{N}^*\times\mathbb{N}^*} \alpha_{n,k} e^{-\alpha n} \sin kx \sin ny / \sqrt{\lambda_{n,k}}\alpha_{n,k} \in l^2 \right\};$$

$$X_{1,\alpha} := \left\{ \sum_{(n,k)\in\mathbb{N}^*\times\mathbb{N}^*} \alpha_{n,k} e^{-\alpha n} \sin kx \sin ny / \frac{\alpha_{n,k}}{\sqrt{\lambda_{n,k}}} \in l^2 \right\};$$

$$X_\alpha := X_{0,\alpha} \times X_{1,\alpha}.$$

Let $T > 2\pi\sqrt{2}$,

$$S_T = \left\{ (u^0, u^1) \in E \,/\, \exists\, C > 0,\, \left\| \frac{\partial[G(\partial_t \phi)]}{\partial\nu} \right\|_{L^2(\Gamma_0\times]0,T[)} \geq C \left\|(u^0, u^1)\right\|_E \right\}.$$

We notice that if $\alpha' > \alpha$, then $X_{\alpha'} \subset X_\alpha$ and $X_0 = E$.

Let $T > 2\pi\sqrt{2}$ and $\underline{u} := (u^0, u^1) \in E$.

For $n \in \mathbb{N}^*$, if we restrict to E_n, there exists $C(n) > 0$ such that for ϕ solution of (3), it holds :

$$\|(u^0, u^1)\|_E \leq C(n) \left\| \frac{\partial [G(\partial_t \phi)]}{\partial \nu} \right\|_{L^2(\Gamma_0 \times]0,T[)}.$$

We take for the $C(n)$ the smallest constant for which the previous inequality is checked and we denote by

$$\alpha_S(T) := \lim \sup_{n \to +\infty} \frac{\ln(C(n))}{n}.$$

According to H.U.M method, see [21], and to [3, chapter 1] we have :

$$\begin{cases} \alpha' > \alpha_S(T) \Longrightarrow X_{\alpha'} \subset S_T, \\ \alpha' < \alpha_S(T) \Longrightarrow X_{\alpha'} \not\subset S_T. \end{cases} \tag{5}$$

Thus for all $T > 2\pi\sqrt{2}$, $\alpha_S(T) = \inf \{\alpha \in \mathbb{R}_+ \,/\, X_\alpha \subset S_T\}$.

We give, now the main results of this paper :

Theorem 2.1 *1. For all $\delta > 0$, there exists a constant $C_\delta > 0$ such that*

$$\alpha_S(T) \leq \frac{C_\delta}{T^{1-\delta}}.$$

2. For $\alpha > \alpha_S(T)$, there exists a constant C_α, $\gamma_\alpha > 0$ such that

$$E(t) \leq C_\alpha \, e^{-\gamma_\alpha t} E(0), \quad \forall \underline{u} \in X_\alpha, \, \forall t \geq 0.$$

Remark 2.1

1. We remark that all the elements of X_α can be continued as an holomorphic function over the complex strip $|\Im m(y)| < \alpha$.
2. The first assertion of the previous theorem implies that any analytic initial condition belongs to some S_T for T large enough, i.e., any initial condition whose Fourier coefficients in y decrase like $e^{-\alpha\, n}$ belongs to S_T if T is larger than $T(\alpha) = \sqrt[1-\delta]{\frac{C_\delta}{\alpha}}$.

Theorem 2.2 *The system described by (1) is polynomial stable i.e., for all $\Gamma_0 \neq \emptyset$, there exists a constant $C > 0$ such as for all $(u^0, u^1) \in \mathcal{D}(\mathcal{A})$ we have :*

$$E(t) \leq \frac{C}{1+t} \, ||(u^0, u^1)||^2_{\mathcal{D}(\mathcal{A})}, \, \forall \, t \geq 0, \tag{6}$$

where

$$\mathcal{D}(\mathcal{A}) = \left\{ (u, v) \in H^1_0(\Omega) \times H^{-1}(\Omega) \,;\, \Delta u \in H^1_0(\Omega), \, \Delta u = \partial_\nu [G(v)] \, 1_{|\Gamma_0} \right\}.$$

3. Inequality of observability

In this section we give the observability inequality at low and high frequency of the solution of (3) has been used for the proof of the main results. We specify the dependence of the constant which occurs in this estimation in function of the frequency of cut n.

Proposition 3.1 *(low frequencies estimate) For all $\epsilon > 0$, $\delta > 0$, there exist $T_1(\epsilon,\delta) \le \frac{C_\delta}{\epsilon^{1+\delta}}$, $C_{\epsilon,\delta}$, $n_1 \in \mathbb{N}^*$ such that for all $n \ge n_1$ and for all $\underline{u} \in E_n$, the solution of problem (3) satisfies*

$$\|\underline{u}\|_{E^1}^2 \le C_{\epsilon,\delta}\, e^{2\epsilon n} \int_0^\pi \int_{-T_1(\epsilon,\delta)}^{T_1(\epsilon,\delta)} \left| \frac{\partial G(\phi')}{\partial x}(0,y,t) \right|^2 dt\, dy.$$

Proof. Since we do not have a uniform gap, we adapt the method proposed by Allibert and Micu in [4], which is a method inspired from the WKB technique. First we need the next technical lemma, for this proof we refer to [4], paragraph 4.3, pages 580-591.

Lemma 3.1 *For all positive and odd integer q and for all $\epsilon > 0$, there exists a positive real number C_q and a real number $T_1(q,\epsilon)$ smaller than $C_{\epsilon,q}\epsilon^{\frac{q+1}{1-q}}$ such that for all $(n,k_0) \in \mathbb{N}^* \times \mathbb{N}^*$, there exists a function $h_{\epsilon,q}^{k_0,n}$ that satisfies :*

1. $\mathrm{supp}(h_{\epsilon,q}^{k_0,n}) \subset [-T_1(q,\epsilon), T_1(q,\epsilon)]$.
2. *For (k_0,n) such that $k_0 \le n$,*
$$\|h_{\epsilon,q}^{k_0,n}\|_{L^2} \le C_{\epsilon,q}\, e^{2\epsilon n}.$$
3. *If $k \ne k_0$, then* $\displaystyle\int h_{\epsilon,q}^{k_0,n}(t) e^{i\lambda_{n,k} t} dt = 0.$
4. *If $(n,k_0) \in \{(n,k) \in \mathbb{N}^* \times \mathbb{N}^* / k \le n \,\text{and}\, n \ge n_1(q,\epsilon)\}$, then*
$$\left| \int h_{\epsilon,q}^{k_0,n}(t) e^{i\lambda_{n,k_0} t} dt \right| \ge \frac{c}{n^{N_q}}.$$

The two above positive constants C, c depend only on q and ϵ. Moreover it is always possible to choose $h_{\epsilon,q}^{k_0,n}$ even or odd, that we denote by $h_{e_\epsilon,q}^{k_0,n}$ and $h_{o_\epsilon,q}^{k_0,n}$.

The analogue of Proposition 3.1 is proved in [4, Lemme 6]. The proof is quite similar, but for the sake of completeness, let us give the main steps.

Proof of Proposition 3.1. Let $n \in \mathbb{N}^*$ be such that $n \ge n_1(q,\epsilon)$, and let

$$(u^0,u^1) = \sum_{k \in \mathbb{N}^*} a_{n,k} \varphi_{n,k} \in E_n.$$

Then we have
$$\frac{\partial G(\phi')}{\partial x}(0,y,t)=\sum_{k\in\mathbb{N}^*}\frac{2i\,k\,a_{n,k}}{\pi\sqrt{\lambda_{n,k}}}e^{i\lambda_{n,k}t}\sin ny.$$

Hence for (k_0,n) such that $k_0\le\ n$ and $L\in\mathbb{N}^*$,

$$\int h^{k_0,n}_{e_\epsilon,q}(t)K\left(\sum_{k\le L}a_{n,k}\varphi_{n,k}\right)dt=$$

$$\sum_{1\le k\le L}\frac{2i\,k}{\pi\sqrt{\lambda_{n,k}}}(a_{n,k}+a_{n,-k})\sin ny\int h^{k_0,n}_{e_\epsilon,q}(t)e^{i\lambda_{n,k}t}dt,$$

where K is the operator defined by

$$K:\qquad \begin{array}{c} E_n\to L^2(0,\pi)\\ (u^0,u^1)\to\frac{\partial[G(\partial_t\phi)]}{\partial x}(0,y,t).\end{array}$$

If $L\ge k_0$, then by the point 3 of Lemma 3.1 we will have

$$\int h^{k_0,n}_{e_\epsilon,q}(t)K\left(\sum_{k\le L}a_{n,k}\varphi_{n,k}\right)(y,t)dt=$$

$$\frac{2i\,k}{\pi\sqrt{\lambda_{n,k_0}}}(a_{n,k_0}+a_{n,-k_0})\sin ny\int h^{k_0,n}_{e_\epsilon,q}(t)e^{i\lambda_{n,k_0}t}dt.$$

For point 4 of Lemma 3.1 we deduce that there exists a constant $c>0$ such that

$$\left|\int h^{k_0,n}_{e_\epsilon,q}(t)K\left(\sum_{k\le L}a_{n,k}\varphi_{n,k}\right)(y,t)dt\right|\ge\frac{c}{n^{N_q+2}}|a_{n,k_0}+a_{n,-k_0}||\sin ny|.$$

Consequently, if L tends to infinity, we obtain

$$\left|\int h^{k_0,n}_{e_\epsilon,q}(t)K\left(\sum_{k\in\mathbb{N}^*}a_{n,k}\varphi_{n,k}\right)(y,t)dt\right|\ge\frac{c}{n^{N_q+2}}|a_{n,k_0}+a_{n,-k_0}||\sin ny|.$$

In the same we obtain

$$\left|\int h^{k_0,n}_{o_\epsilon,q}(t)K\left(\sum_{k\in\mathbb{N}^*}a_{n,k}\varphi_{n,k}\right)(y,t)dt\right|\ge\frac{c}{n^{N_q+2}}|a_{n,k_0}-a_{n,-k_0}||\sin ny|.$$

These two estimates yield

$$|\sin ny||a_{n,k_0}|\le\frac{n^{N_q}}{c}\left(\left|\int h^{k_0,n}_{e_\epsilon,q}(t)K\left(\sum_{k\in\mathbb{N}^*}a_{n,k}\varphi_{n,k}\right)(y,t)dt\right|+\right.$$

$$\frac{n^{N_q}}{c}\left|\int h^{k_0,n}_{o_\epsilon,q}(t)K\left(\sum_{k\in\mathbb{N}^*} a_{n,k}\varphi_{n,k}\right)(y,t)dt\right|\Bigg).$$

Then

$$|\sin ny|^2\|(u_0,u_1)\|_E^2 \leq \frac{n^{N_q+2}}{c}\left(\sum_{k\leq n}\int \left|h^{k_0,n}_{e_\epsilon,q}(t)\right|^2\int_{T_1(q,\epsilon)}^{T_1(q,\epsilon)}\left|\frac{\partial[G(\phi')]}{\partial x}(0,y,t)\right|^2 dt+\right.$$

$$\left.\sum_{k\leq n}\int \left|h^{k_0,n}_{o_\epsilon,q}(t)\right|^2\int_{T_1(q,\epsilon)}^{T_1(q,\epsilon)}\left|\frac{\partial[G(\phi')]}{\partial x}(0,y,t)\right|^2 dt\right).$$

Integrating this estimate in $y\in(0,\pi)$ and using point 2 of Lemma 3.1, we obtain a constant $c_1>0$ such that

$$\|(u_0,u_1)\|_E^2 \leq c_1\, e^{2\epsilon n}\int_{T_1(q,\epsilon)}^{T_1(q,\epsilon)}\int_0^\pi\left|\frac{\partial[G(\phi')]}{\partial x}(0,y,t)\right|^2 dy\,dt.$$

As $T_1(q,\varepsilon)\leq C_q\varepsilon^{\frac{1+q}{1-q}}=\frac{C_\delta}{\varepsilon^{1+\delta}}$ and $\delta\to 0^+$ for $q\to+\infty$, this shows Proposition 3.1. □

Lemma 3.2 *(High frequencies) For all $T_2>2\pi\sqrt{2}$, there exists a constant $C_{T_2}>0$ such that for all integer $n>0$ and initial data $\underline{u}$ in E_n^2 the solution of problem (3) satisfies*

$$\|\underline{u}\|^2_{E^2}\leq C_{T_2}\int_0^\pi\int_0^{T_2}\left|\frac{\partial[G(\partial_t\phi)]}{\partial x}(0,y,t)\right|^2 dt\,dy.$$

Proof. For $\underline{u}\in E_n^2$, we have $\partial_t\phi(x,y,t)=\sum_{k>n}\frac{2a_{n,k}}{\pi}\sqrt{\lambda_{n,k}}\sin kx\sin ny\, e^{it\lambda_{n,k}}$. Then if we use the Ingham inequality [11], we obtain

$$\begin{aligned}\int_0^\pi\int_0^{T_2}\left|\frac{\partial[G(\partial_t\phi)]}{\partial x}(0,y,t)\right|^2 dtdy &= \sum_{n\in\mathbb{N}^*}\frac{1}{\pi^2}\int_0^{T_2}\left|\sum_{k>n}\frac{ka_{n,k}}{\sqrt{\lambda_{n,k}}}e^{i\lambda_{n,k}t}dt\right|^2\\ &\geq C_{T_2}\sum_{k>n}\left|\frac{ka_{n,k}}{\sqrt{\lambda_{n,k}}}\right|^2.\end{aligned}$$

Which implies $\int_0^\pi\int_0^{T_2}\left|\frac{\partial[G(\partial_t\phi)]}{\partial x}(0,y,t)\right|^2 dtdy\geq C_{T_2}\|\underline{u}\|^2_{E^2}$. □

4. Some background on a class of dynamical systems

Let H a Hilbert space with the norm $||.||_H$, and let $A_1 : \mathcal{D}(A_1) \to H$ be a self-adjoint, positive and boundedly invertible operator. For $\alpha \geq 0$, we introduce the scale of Hilbert spaces $H_\alpha = \mathcal{D}(A_1^\alpha)$, with the norm $\|z\|_\alpha = \|A_1^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows : $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. The operator A_1 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_1 : H_\alpha \to H_{\alpha-1}, \quad \forall\, \alpha \in \mathbb{R}. \tag{7}$$

The second ingredient needed for our construction is a bounded linear operator $B_1 : U \to H_{-\frac{1}{2}}$, where U is another Hilbert space which will be identified with its dual.

The system we consider are described by

$$\ddot{w}(t) + A_1 w(t) + B_1 y(t) = 0,\, w(0) = w_0,\, \dot{w}(0) = w_1,\, t \in [0, \infty), \tag{8}$$

$$y(t) = B_1^* \dot{w}(t),\, t \in [0, \infty). \tag{9}$$

The system (8)-(9) is well-posed :
For $(w_0, w_1) \in H_{\frac{1}{2}} \times H$, the problem (8)-(9) allows a unique solution :

$$w \in C([0, \infty); H_{\frac{1}{2}}) \cap C^1([0, \infty); H)$$

such that $B_1^* w(\cdot) \in H^1(0, T; U)$. Moreover, w satisfies the energy estimate, for all $t \geq 0$:

$$\|(w_0, w_1)\|^2_{H_{\frac{1}{2}} \times H} - \|(w(t), \dot{w}(t))\|^2_{H_{\frac{1}{2}} \times H} = 2 \int_0^t \left\| \frac{d}{dt} B_1^* w(s) \right\|_U^2 ds\,. \tag{10}$$

For (10) we remark that the mapping $t \mapsto \|(w(t), \dot{w}(t))\|^2_{H_{\frac{1}{2}} \times H}$ is non-increasing.
Consider the initial value problem:

$$\ddot{\varphi}(t) + A_1 \varphi(t) = 0, \tag{11}$$

$$\varphi(0) = w_0,\, \dot{\varphi}(0) = w_1. \tag{12}$$

It is well known that (11)-(12) is well posed in $H_1 \times H_{\frac{1}{2}}$ and in $H_{\frac{1}{2}} \times H$.
Now, we consider the unbounded linear operator

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \to H_{\frac{1}{2}} \times H,\, \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -A_1 & -B_1 B_1^* \end{pmatrix}, \tag{13}$$

where

$$\mathcal{D}(\mathcal{A}_d) = \Big\{ (u, v) \in H_{\frac{1}{2}} \times H,\, A_1 u + B_1 B_1^* v \in H,\, v \in H_{\frac{1}{2}} \Big\}.$$

The result below, proved in [5], shows that, under a certain regularity assumption, the exponential stability of (8)-(9) is equivalent to a strong observability inequality for (11)-(12) and the polynomial stability of (8)-(9) is a consequence of weak observability inequality. More precisely, we have :

Theorem 4.1 (Ammari-Tucsnak [5]) *Assume that for any $\gamma > 0$ we have*

$$\sup_{\mathrm{Re}\lambda=\gamma} \left\| \lambda B_1^*(\lambda^2 I + A_1)^{-1} B_1 \right\|_{\mathcal{L}(U)} < \infty. \tag{14}$$

Then

1. *there exists $C, \delta > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in H_{\frac{1}{2}} \times H$, we have*
$$\|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}}\times H} \leq C\, e^{-\delta t}\, ||(w^0, w^1)||_{H_{\frac{1}{2}}\times H},$$
if and only if there exists $T, C > 0$ such that : $\forall\, (w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$, we have
$$||B_1^* \varphi'(t)||_{L^2(0,T;U)} \geq C\; ||(w_0, w_1)||_{H_{\frac{1}{2}}\times H}\,, \tag{15}$$
where $\varphi(t)$ is the solution of system (11)-(12).

2. *If there exists $T, C > 0$ and $\alpha > -\frac{1}{2}$ such that : $\forall\, (w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$, we have*
$$||B_1^* \varphi'(t)||_{L^2(0,T;U)} \geq C\; ||(w_0, w_1)||_{H_{-\alpha}\times H_{-\alpha-\frac{1}{2}}}\,, \tag{16}$$
where $\varphi(t)$ is the solution of (11)-(12).

   Then there exists a constant $C_1 > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in \mathcal{D}(\mathcal{A}_d)$, we have
$$\|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}}\times H} \leq \frac{C_1}{(1+t)^{\frac{1}{4\alpha+2}}}\, ||(w^0, w^1)||_{\mathcal{D}(\mathcal{A}_d)}. \tag{17}$$

5. Proof of the main results

5.1. Proof of the first assertion of the Theorem 2.1. For this proof, we need a result of the following lemma inspired by [19] (see also [7]).

Lemma 5.1 *Let $v \in E$ then $v \in S_T$ if and only if there exists a constant $C_v > 0$ such that for any initial data $\underline{u} \in E$, the solution u of problem (3) satisfies*

$$|\langle \underline{u}, v \rangle_E| \geq C_v \left\| \frac{\partial [G(\partial_t u)]}{\partial \nu}(., t) \right\|_{L^2(\Gamma_0 \times (0,T))}.$$

Then, let $\delta,\ \epsilon > 0$ and $v \in X_\epsilon$. We can put $v(x, y) = \sum_{n\in\mathbb{N}^*} e^{-\epsilon n} v^n(x) \sin ny$, with $(\|v^n\|_E)_n \in l^2(\mathbb{N}^*)$. Take $T(\epsilon, \delta) = \sup\,\left(T_1(\epsilon, \delta), 2\pi\sqrt{2}\right)$. As $T_1(\epsilon, \delta) \leq \frac{C}{\epsilon^{1+\delta}}$, for small ϵ, then $T(\epsilon, \delta) \leq \frac{C}{\epsilon^{1+\delta}}$. For any $\underline{u} \in E$ and $\underline{v} = (0, v)$, we have

$$|\langle \underline{u}, \underline{v} \rangle_E| = \left| \sum_{n\in\mathbb{N}^*} \langle e^{-\epsilon n} \underline{u}^n, \underline{v}^n \sin ny \rangle_E \right|.$$

We deduce

$$|\langle \underline{u}, \underline{v}\rangle_E| \leq \sum_{n\in\mathbb{N}^*} e^{-\epsilon n}\|\underline{u}^n\|_E\|v^n \sin ny\|_E. \tag{18}$$

As $\forall\ n \in \mathbb{N}^*$, and $\forall\ \underline{u}^n \in E_n$, we have

$$\|\underline{u}\|_E^2 = \|\underline{u}\|_{E^1}^2 + \|\underline{u}\|_{E^2}^2.$$

According to Proposition 3.1 and Lemma 3.2

$$\|\underline{u}\|_E^2 \leq C_{\epsilon,\delta}\, e^{2\epsilon n}\int_0^\pi \int_{-T_1(\epsilon,\delta)}^{T_1(\epsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dt\,dy +$$

$$C_T \int_0^\pi \int_0^{T(\varepsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dt\,dy.$$

Then

$$\|\underline{u}\|_E^2 \leq C'_{\epsilon,\delta}\, e^{2\epsilon n}\int_0^\pi \int_{-T(\epsilon,\delta)}^{T(\epsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dy\,dt +$$

$$C_T \int_0^\pi \int_0^{T(\epsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dt\,dy + C_T \int_0^\pi \int_0^{T(\epsilon,\delta)} \left|\frac{\partial[G(u'^1)]}{\partial x}(0,y,t)\right|^2 dt\,dy.$$

Which implies

$$\|\underline{u}\|_E^2 \leq C'_{\epsilon,\delta} e^{2\epsilon n}\int_0^\pi \int_{-T(\epsilon,\delta)}^{T(\epsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dt\,dy +$$

$$C_T \int_0^\pi \int_0^{T(\epsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dt\,dy + C'\|\underline{u}\|_{E^1}^2.$$

This inequality according to Proposition 3.1 implies

$$\|\underline{u}\|_E^2 \leq C''_{\epsilon,\delta} e^{2\epsilon n}\int_0^\pi \int_{-T(\epsilon,\delta)}^{T(\epsilon,\delta)} \left|\frac{\partial[G(u')]}{\partial x}(0,y,t)\right|^2 dt\,dy.$$

If we replace this estimate in (18), we obtain

$$|\langle \underline{u}, v\rangle_E| \leq C''_{\epsilon,\delta} \sum_{n\in\mathbb{N}^*} \|v^n(x)\sin ny\|_E \sqrt{\int_0^\pi \int_0^{2T(\epsilon,\delta)} \left|\frac{\partial[G(\partial u^n/\partial_t)]}{\partial x}(0,y,t)\right|^2 dt\,dy}$$

$$\leq C''_{\epsilon,\delta}\sqrt{\sum_{n\in\mathbb{N}^*}\int_0^\pi \int_0^{2T(\epsilon,\delta)} \left|\frac{\partial[G(\partial u^n/\partial_t)]}{\partial x}(0,y,t)\right|^2 dt\,dy}\sqrt{\sum_{n\in\mathbb{N}^*}\|v^n(x)\sin ny\|_E^2}$$

$$\leq C''_{\epsilon,\delta} C_v \left\|\frac{\partial[G(u'^n)]}{\partial \nu}\right\|_{L^2(\Gamma_0\times(0,2T(\epsilon,\delta))}.$$

This implies that $v \in S_T$, i.e $X_\epsilon \subset S_T$, as well as $\alpha_S(T) \leq \frac{C_\delta}{T^(1-\delta)}$.

□

5.2. PROOF OF THE SECOND ASSERTION OF THEOREM 2.1. For $\alpha > \alpha_S(T)$ we have $X_\alpha \subset S_T$. Then, there exists a constant $C > 0$ such that the solution ϕ of problem (3) satisfies

$$\int_0^T \int_0^\pi \left| \frac{\partial [G(\partial\phi/\partial t)]}{\partial x} \right| d\,y\,d\,t \geq C(T)E(0), \qquad \forall (u^0, u^1) \in X_\alpha.$$

This inequality according to [5, Theorem 2.2] (see also Theorem 4.1) implies the existence of C_α, $\delta_\alpha > 0$ such that

$$E(t) \leq C_\alpha e^{-\gamma_\alpha t} E(0), \qquad \forall\, (u^0, u^1) \in X_\alpha,\, \forall\, t \geq 0.$$

5.3. PROOF OF THEOREM 2.2. Let $\underline{u} \in E$, we have

$$\underline{u} = \sum_{n,k \geq 1} a_{n,k} \varphi_{n,k},$$

where $(a_{n,k}) \in l^2$. Then

$$\partial_t \phi(x, y, t) = \sum_{n,k \in \mathbb{Z}^*} \frac{2a_{n,k}}{\pi} \sqrt{\lambda_{n,k}} \sin kx \sin ny\, e^{it\lambda_{n,k}}.$$

Due to the orthogonality of the family $(\sin ny)$ in $L^2(0, \pi)$, we get for $T > 2\pi\sqrt{2}$

$$\int_0^\pi \int_0^T \left| \frac{\partial [G(\partial_t \phi)]}{\partial x}(0, y, t) \right|^2 dtdy = \sum_{n \in \mathbb{Z}^*} \frac{1}{\pi^2} \int_0^T \left| \sum_{k \in \mathbb{Z}^*} \frac{k a_{n,k}}{\sqrt{\lambda_{n,k}}} e^{i\lambda_{n,k} t} dt \right|^2.$$

According to [22, Theorem 2.1], we obtain the existence of constant $C_2 > 0$ satisfies

$$\int_0^T \int_0^\pi \left| \frac{\partial [G(\partial_t \phi)]}{\partial x}(0, y, t) \right|^2 dy\, dt \geq C_2 \sum_{n,k \in \mathbb{Z}^*} \left| \frac{a_{n,k}}{\sqrt{\lambda_{n,k}}} \right|^2 \sim C_2 \left\| \underline{u} \right\|^2_{H^{-1}(\Omega) \times [H_0^1(\Omega)]'},$$

where $[H_0^1(\Omega)]'$ is the dual space of $H_0^1(\Omega)$. The duality is respected to the pivot space $H^{-1}(\Omega)$.

Which shows Theorem 2.2 thanks to Theorem 4.1 for $\alpha = 0$.

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