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On (δ, p)-continuous functions and (δ, p)-closed graphs

M. Caldas, E. Ekici, S. Jafari and S. P. Moshokoa

ABSTRACT: It is the object of this paper to introduce the notions of (δ, p) -continuity and (δ, p)-closed graphs by utilizing the notion of (δ, p)-open sets and investigate the fundamental properties of (δ, p)-continuous functions and also present some properties of functions with (δ, p)-closed graphs.

Key Words: Topological spaces, (δ, p)-open set, (δ, p)-closed graph, (δ, p)-T_1, (δ, p)-continuous. (δ, p)-W-continuous.

Contents

1 Introduction **9**

2 Some properties **9**

1. Introduction

In this paper X and Y denote the topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively. Jafari [2] introduced the notion of pre-regular p-open sets and further investigated its fundamental properties in [3]. A subset A of a topological space (X, τ) is called a *pre-regular p-open* [2] if $A = pInt(pCl(A))$. Now we recall the following notions from [1] which will be used in the sequel: A point $x \in X$ is called the (δ, p)-*cluster point* of A if $A \cap U \neq \emptyset$ for every pre-regular p-open set U of X containing x. The set of all (δ, p)-cluster points of A is called the (δ, p)-*closure* of A, denoted by $\delta Cl_p(A)$. If $\delta Cl_p(A) = A$, then A is called (δ, p)-*closed*. The complement of a (δ, p)-closed set is called (δ, p)-*open*. We say that a set U in a topological space (X, τ) is a (δ, p)-*neighborhood* of a point x if U contains a (δ, p)-open set to which x belongs. We denote the collection of all (δ, p)-open (respectively (δ, p)-closed) sets by $\delta PO(X, \tau)$ (respectively $\delta PC(X, \tau)$).

In this paper we offer a new class of functions called (δ, p)-continuous functions and a new notion of the graph of a function called a (δ, p)-closed graph. We also investigate some of their fundamental properties.

2. Some properties

Definition 2.1 *A function $f : X \to Y$ is said to be (δ, p)-continuous if for every open set V of Y, $f^{-1}(V)$ is (δ, p)-open in X.*

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Theorem 2.1 *The following are equivalent for a function $f : X \to Y$:*
(1) f is (δ, p)-continuous,
(2) The inverse image of every closed set in Y is (δ, p)-closed in X,
(3) For each subset A of X, $f(\delta Cl_p(A)) \subset Cl(f(A))$,
(4) For each subset B of Y, $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

Proof. (1) $\Leftrightarrow$ (2) : Obvious.

(3) $\Leftrightarrow$ (4) : Let B is any subset of Y. Then by (3), we have $f(\delta Cl_p(f^{-1}(B))) \subset Cl(f(f^{-1}(B))) \subset Cl(B)$. This implies $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(f(\delta Cl_p(f^{-1}(B)))) \subset f^{-1}(Cl(B))$.

Conversely, let $B = f(A)$ where A is a subset of X. Then, by (4), we have, $\delta Cl_p(A) \subset \delta Cl_p(f^{-1}(f(A))) \subset f^{-1}(Cl(f(A)))$. Thus, $f(\delta Cl_p(A)) \subset Cl(f(A))$.

(2) $\Rightarrow$ (4) : Let $B \subset Y$. Since $f^{-1}(Cl(B))$ is (δ, p)-closed and $f^{-1}(B) \subset f^{-1}(Cl(B))$, then $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

(4) $\Rightarrow$ (2) : Let $K \subset Y$ be a closed set. By (4), $\delta Cl_p(f^{-1}(K)) \subset f^{-1}(Cl(K)) = f^{-1}(K)$. Thus, $f^{-1}(K)$ is (δ, p)-closed.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 2.2 *For a function $f : X \to Y$, the graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is said to be (δ, p)-closed if for each $(x, y) \in X \times Y \setminus G(f)$, there exist $U \in \delta PO(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 2.1 *Let $f : X \to Y$ be a function. Then the graph $G(f)$ is (δ, p)-closed in $X \times Y$ if and only if for each point $(x, y) \in X \times Y \setminus G(f)$, there exist a (δ, p)-open set U and an open set V containing x and y, respectively, such that $f(U) \cap V = \emptyset$.*

Proof. It follows readily from the above definition.

Definition 2.3 *A space X is said to be (δ, p)-T_1 [1] if for each pair of distinct points x and y of X, there exist a (δ, p)-open set U containing x but not y and a (δ, p)-open set V containing y but not x.*

Theorem 2.2 *If $f : X \to Y$ is an injective function with the (δ, p)-closed graph, then X is (δ, p)-T_1.*

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Thus there exist a (δ, p)-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \emptyset$. Therefore $y \notin U$ and it follows that X is (δ, p)-T_1.

Recall that a space X is said to be T_1 if for each pair of distinct points x and y of X, there exist an open set U containing x but not y and an open set V containing y but not x.

Theorem 2.3 *If $f : X \to Y$ is a surjective function with the (δ, p)-closed graph, then Y is T_1.*

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists a point x in X such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By Lemma 2.1, there exist a (δ, p)-open set U and an open set V containing x and y_1, respectively, such that $f(U) \cap V = \emptyset$. It follows that $y_2 \notin V$. Hence Y is T_1.

Definition 2.4 *A function $f : X \to Y$ is said to be (δ, p)-W-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a (δ, p)-open set U in X containing x such that $f(U) \subset Cl(V)$.*

Theorem 2.4 *If $f : X \to Y$ is (δ, p)-W-continuous and Y is Hausdorff, then $G(f)$ is (δ, p)-closed.*

Proof. Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that Y is Hausdorff, there exist open sets W and V such that $f(x) \in W$, $y \in V$ and $V \cap W = \emptyset$. It follows that $Cl(W) \cap V = \emptyset$. Since f is (δ, p)-W-continuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset Cl(W)$. Hence, we have $f(U) \cap V = \emptyset$. This means that $G(f)$ is (δ, p)-closed.

Corollary 2.4A *If $f : X \to Y$ is (δ, p)-continuous and Y is Hausdorff, then $G(f)$ is (δ, p)-closed in $X \times Y$.*

Definition 2.5 *A subset A of a space X is said to be (δ, p)-compact relative to X if every cover of A by (δ, p)-open sets of X has a finite subcover.*

Theorem 2.5 *Let $f : X \to Y$ have a (δ, p)-closed graph. If K is (δ, p)-compact relative to X, then $f(K)$ is closed in Y.*

Proof. Suppose $y \notin f(K)$. For each $x \in K$, $f(x) \neq y$. By Lemma 2.1, there exist $U_x \in \delta PO(X, x)$ and an open neighbourhood V_x of y such that $f(U_x) \cap V_x = \emptyset$. The family $\{U_x \mid x \in K\}$ is a cover of K by (δ, p)-open sets of X and there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x \mid x \in K_0\}$. Put $V = \bigcap\{V_x \mid x \in K_0\}$. Then V is an open neighbourhood of y and $f(K) \cap V = \emptyset$. This means that $f(K)$ is closed in Y.

Definition 2.6 *A function $f : X \to Y$ is called perfectly continuous [4] if for each open set $A \subset Y$, $f^{-1}(A)$ is open and closed in X.*

Lemma 2.2 *([3]) If A and B are pre-regular p-open sets of the spaces X and Y, respectively, then $A \times B$ is a pre-regular p-open set of $X \times Y$.*

Theorem 2.6 *If $f : X \to Z$ has a (δ, p)-closed graph $G(f)$ and $g : Y \to Z$ is a perfectly continuous function, then the set $\{(x, y) : f(x) = g(y)\}$ is (δ, p)-closed in $X \times Y$.*

Proof. Let $A = \{(x, y) : f(x) = g(y)\}$ and $(x, y) \in X \backslash A$. We have $f(x) \neq g(y)$ and then $(x, g(y)) \in (X \times Z) \backslash G(f)$. Since f has a (δ, p)-closed graph $G(f)$, there exist a (δ, p)-open set U and an open set V containing x and $g(y)$, respectively such that $f(U) \cap V = \varnothing$. This implies that there exists a pre-regular p-open set N containing x such that $N \subset U$ and $f(N) \cap V = \varnothing$. Since g is a perfectly continuous function, then there exist an open and closed set G containing y such that $g(G) \subset V$. We have $f(U) \cap g(G) = \varnothing$. This implies that $(N \times G) \cap A = \varnothing$. Since $N \times G$ is pre-regular p-open, then $(x, y) \notin \delta Cl_p(A)$. Thus, E is (δ, p)-closed in $X \times Y$.

Corollary 2.6B *If $f : X \to Z$ is a (δ, p)-continuous function and $g : Y \to Z$ is a perfectly continuous function and Z is Hausdorff, then the set $\{(x, y) : f(x) = g(y)\}$ is (δ, p)-closed in $X \times Y$.*

Proof. It follows from Corollary 2.6A and Theorem 2.6.

Theorem 2.7 *If $f : X \to Y$ is a (δ, p)-continuous function and Y is Hausdorff, then the set $\{(x, y) \in X \times X : f(x) = f(y)\}$ is (δ, p)-closed in $X \times X$.*

Proof. Let $A = \{(x, y) : f(x) = f(y)\}$ and let $(x, y) \in (X \times X)\backslash A$. It follows that $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open sets U and V containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \emptyset$. Since f is (δ, p)-continuous, there exist pre-regular p-open sets H and G in X containing x and y, respectively, such that $f(H) \subset U$ and $f(G) \subset V$. This implies $(H \times G) \cap A = \emptyset$. We have $H \times G$ is a pre-regular p-open set in $X \times X$ containing (x, y). Hence, A is (δ, p)-closed in $X \times X$.

Definition 2.7 *A function $f : X \to Y$ is called contra (δ, p)-open if the image of every (δ, p)-open set in X is closed in Y.*

Theorem 2.8 *If $f : X \to Y$ is a contra (δ, p)-open function such that inverse image of each point of Y is (δ, p)-closed, then f has a (δ, p)-closed graph $G(f)$.*

Proof. Let $(x, y) \in X\backslash G(f)$. We have $x \notin f^{-1}(y)$. Since $f^{-1}(y)$ is (δ, p)-closed, there exists a pre-regular p-open set A containing x such that $A \cap f^{-1}(y) = \varnothing$. Since f is contra (δ, p)-open, then $f(A)$ is closed. This implies that there exist an open set B in Y containing y such that $f(A) \cap B = \varnothing$. Hence, f has a (δ, p)-closed graph $G(f)$.

Theorem 2.9 *If $f : (X, \tau) \to (Y, \sigma)$ has a (δ, p)-closed graph $G(f)$, then for each $x \in X$, $\{f(x)\} = \bigcap_{x \in A \in \delta PO(X,\tau)} Cl(f(A))$.*

Proof. Suppose that $y \neq f(x)$ and $y \in \bigcap_{x \in A \in \delta PO(X,\tau)} Cl(f(A))$. Then $y \in Cl(f(A))$ for each $x \in A \in \delta PO(X, \tau)$. This implies that for each open set B containing y, $B \cap f(A) \neq \varnothing$. Since $(x, y) \notin G(f)$ and $G(f)$ is a (δ, p)-closed graph, this is a contradiction.

Definition 2.8 *A space X is said to be (δ, p)-T_2 if for each pair of distinct points x and y in X, there exist disjoint (δ, p)-open sets A and B in X such that $x \in A$ and $y \in B$.*

Definition 2.9 *A function $f : X \to Y$ is called (δ, p)-open if the image of every (δ, p)-open set in X is open in Y.*

Theorem 2.10 *If $f : X \to Y$ is a surjective (δ, p)-open function with a (δ, p)-closed graph $G(f)$, then Y is T_2 .*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y)\backslash G(f)$. This implies that there exist a (δ, p)-open set A of X and an open set B of Y such that $(x, y_2) \in A \times B$ and $(A \times B) \cap G(f) = \emptyset$. We have $f(A) \cap B = \emptyset$. Since f is (δ, p)-open, then $f(A)$ is open such that $f(x) = y_1 \in f(A)$. Thus, Y is T_2.

Theorem 2.11 *If $f : X \to Y$ is a (δ, p)-continuous injection and Y is T_2, then X is (δ, p)-T_2.*

Proof. Let x and y in X be any pair of distinct points. Then there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. Since f is (δ, p)-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ is (δ, p)-open in X containing x and y respectively. We have $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus, X is (δ, p)-T_2.

Lemma 2.3 *([3]) If a space X is submaximal, then any finite intersection of pre-regular p-open sets is pre-regular p-open.*

Theorem 2.12 *If f, $g : X \to Y$ are (δ, p)-continuous functions, X is submaximal and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is (δ, p)-closed in X.*

Proof. Let $A = \{x \in X : f(x) = g(x)\}$. Take $x \in X\backslash A$. We have $f(x) \neq g(x)$. Since Y is Hausdorff, then there exist open sets U and V in Y containing $f(x)$ and $g(x)$, respectively, such that $U \cap V = \emptyset$. Since f and g are (δ, p)-continuous, then $f^{-1}(U)$ and $g^{-1}(V)$ are (δ, p)-open in X with $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$. Then there exist pre-regular p-open sets G and H such that $x \in G \subset f^{-1}(U)$ and $x \in H \subset g^{-1}(V)$. Take $K = G \cap H$. By Lemma 2.3, K is pre-regular p-open. Thus, $f(K) \cap g(K) = \emptyset$ and hence $x \notin \delta Cl_p(A)$. This shows that A is (δ, p)-closed in X.

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M. Caldas
Departamento de Matematica Aplicada,
Universidade Federal Fluminense,
Rua Mario Santos Braga, SN
CEP. 24020-140, Niteroi, RJ BRASIL.
gmamccs@vm.uff.br

and

E. Ekici
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus, 17020 Canakkale, Turkey.
eekici@comu.edu.tr

and

S. Jafari
Departmento of Economics
Copenhagen University
Oester Farimagsgade 5, Building 26
1353 – Copnehagen K, Denmark.
jafari@stofanet.dk

and

S. P. Moshokoa
Department of Mathematics
University of South Africa
Pretoria 0003, SOUTH AFRICA.
moshosp@unisa.ac.za