Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/spm

(3s.) **v. 27** 1 (2009): 9–14.
ISSN-00378712 in press
doi:10.5269/bspm.v27i1.9061

# The existence result for a system of generalized mixed quasi-variational inclusions with $(H, \eta)$-monotone operators*

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ABSTRACT: In this paper, we prove a new existence result for a system of generalized set-valued quasi-variational inclusions by using a fixed point technique.

Key Words: System of generalized mixed quasi-variational inclusions; $(H, \eta)$-monotone operator; Fixed point technique; Solution existence

## Contents



## 1. Introduction

In this paper, we will use the notations and definitions used in [1]. We suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, $H_1, g_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$, $H_2, g_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$, $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \longrightarrow \mathcal{H}_1$, $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$, $F, P : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1$, $G, Q : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ are all single-valued mappings and $A, C : \mathcal{H}_1 \longrightarrow CB(\mathcal{H}_1)$, $B, D : \mathcal{H}_2 \longrightarrow CB(\mathcal{H}_2)$ are four set-valued mappings. Let $M : \mathcal{H}_1 \longrightarrow 2^{\mathcal{H}_1}$ be an $(H_1, \eta_1)$-monotone operator and $N : \mathcal{H}_2 \longrightarrow 2^{\mathcal{H}_2}$ be an $(H_2, \eta_2)$-monotone operator. We consider the following problem of finding $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{cases} 0 \in F(x,y) + P(u,v) + M(g_1(x)), \\ 0 \in G(x,y) + Q(w,z) + N(g_2(y)). \end{cases} \tag{1.1}$$

Where $u \in A(x), v \in B(y), w \in C(x), z \in D(y)$.

The problem (1.1) is called a system of generalized mixed quasi-variational inclusions and was introduced and studied by Peng and Zhu in [1]. By [1], it is easy to see that problem (1.1) is an important mathematical model and contains some systems of variational inclusions and systems of variational inequalities as special cases.

The main result obtained by Peng and Zhu [1] can be stated as follows:

**Theorem A.** For $i = 1, 2$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be Lipshitz continuous with constant $\tau_i$, $H_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be strongly $\eta_i$-monotone and Lipschitz continuous with constant $\gamma_i$ and $\delta_i$, respectively, $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be strongly monotone and

* This research was partially supported by the Natural Science Foundation of China (Grant No. 10171118), the Science and Technology Research Project of Chinese Ministry of Education (Grant No.206123), the Education Committee project Research Foundation of Chongqing (Grant No. 030801), and the Science Committee project Research Foundation of Chongqing (Grant No. 8409).

2000 *Mathematics Subject Classification:* 49J40; 90C33

Lipschitz continuous with constant $r_i$ and $s_i$, respectively. Let $A, C : \mathcal{H}_1 \longrightarrow CB(\mathcal{H}_1)$, $B, D : \mathcal{H}_2 \longrightarrow CB(\mathcal{H}_2)$ be $\tilde{D}$-Lipschitz continuous with constants $l_A > 0$, $l_C > 0$, $l_B > 0$, and $l_D > 0$, respectively. Let $F : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ be strongly monotone with respect to $\hat{g}_1$ in the first argument with constant $\alpha_1 > 0$, Lipschitz continuous in the first argument with constant $\beta_1 > 0$, and Lipschitz continuous in the second argument with constant $\xi_1 > 0$, respectively, where $\hat{g}_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is defined by $\hat{g}_1(x) = H_1 \circ g_1(x) = H_1(g_1(x)), \forall x \in \mathcal{H}_1$. Let $G : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ be strongly monotone with respect to $\hat{g}_2$ in the second argument with constant $\alpha_2 > 0$, Lipschitz continuous in the second argument with constant $\beta_2 > 0$, and Lipschitz continuous in the first argument with constant $\xi_2 > 0$, respectively, where $\hat{g}_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is defined by $\hat{g}_2(y) = H_2 \circ g_2(y) = H_2(g_2(y)), \forall y \in \mathcal{H}_2$. Assume that $P : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ is Lipschitz continuous in the first and second argument with constants $\mu_1 > 0$ and $\nu_1 > 0$, respectively, $Q : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is Lipschitz continuous in the first and second argument with constants $\mu_2 > 0$ and $\nu_2 > 0$, respectively, $M : \mathcal{H}_1 \longrightarrow 2^{\mathcal{H}_1}$ is an $(H_1, \eta_1)$-monotone operator and $N : \mathcal{H}_2 \longrightarrow 2^{\mathcal{H}_2}$ is an $(H_2, \eta_2)$-monotone operator.

If there exist constants $\lambda > 0$ and $\rho > 0$ such that

$$\begin{cases} \sqrt{1 - 2r_1 + {s_1}^2} + \frac{\tau_1}{\gamma_1}(\sqrt{{\delta_1}^2 {s_1}^2 - 2\lambda\alpha_1 + \lambda^2 {\beta_1}^2} + \lambda\mu_1 l_A) + \rho(\xi_2 + \mu_2 l_C)\frac{\tau_2}{\gamma_2} < 1, \\ \sqrt{1 - 2r_2 + {s_2}^2} + \frac{\tau_2}{\gamma_2}(\sqrt{{\delta_2}^2 {s_2}^2 - 2\rho\alpha_2 + \rho^2 {\beta_2}^2} + \rho\nu_2 l_D) + \lambda(\xi_1 + \nu_1 l_B)\frac{\tau_1}{\gamma_1} < 1. \end{cases}$$

Then problem (1.1) admits a solution $(x, y)$ with $u \in A(x), v \in B(y), w \in C(x), z \in D(y)$ and sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ converge to $x, y, u, v, w, z$, respectively, where $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ are the sequences generated by Algorithm 3.1 in [1].

By Lemma 3.1 in [1], we can easily get the following result:

**Lemma 1.1.** Let $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$; $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \to \mathcal{H}_2$ be two single-valued operators. $H_1 : \mathcal{H}_1 \to \mathcal{H}_1$ be a strictly $\eta_1$-monotone operator and $H_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be a strictly $\eta_2$-monotone operator and $M : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be an $(H_1, \eta_1)$-monotone operator, $N : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be an $(H_2, \eta_2)$-monotone operator. Then the following statements are equivalent each other:

(a) $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a solution of problem (1.1);

(b) $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ satisfies

$$\begin{cases} g_1(x) = R^{H_1,\eta_1}_{M,\lambda}[H_1(g_1(x)) - \lambda F(x, y) - \lambda P(u, v)], \\ g_2(y) = R^{H_2,\eta_2}_{N,\rho}[H_2(g_2(y)) - \rho G(x, y) - \rho Q(w, z)]\}. \end{cases} \tag{1.2}$$

Where $u \in A(x), v \in B(y), w \in C(x), z \in D(y)$ and $R^{H_1,\eta_1}_{M,\lambda} = (H_1 + \lambda M)^{-1}$, $R^{H_2,\eta_2}_{N,\rho} = (H_2 + \rho N)^{-1}$, $\lambda > 0$ and $\rho > 0$ are constants;

(c) The set-valued mapping $E : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ defined by

$$\begin{cases} E(y) = \bigcup_{w \in C(x)} \bigcup_{z \in D(y)} [y - g_2(y) + R^{H_2,\eta_2}_{N,\rho}(H_2(g_2(y)) - \rho G(x, y) - \rho Q(w, z))], \\ g_1(x) = R^{H_1,\eta_1}_{M,\lambda}[H_1(g_1(x)) - \lambda F(x, y) - \lambda P(u, v)]\}, where \quad u \in A(x), v \in B(y) \end{cases} \tag{1.3}$$

has a fixed point in $\mathcal{H}_2$.

**Lemma 1.2 [2].** Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued Lipschitz continuous operator with constant $\tau$, $H : \mathcal{H} \longrightarrow \mathcal{H}$ be a strongly $\eta$-monotone operator with constant $\gamma > 0$ and $M : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be an $(H, \eta)$-monotone operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : \mathcal{H} \longrightarrow \mathcal{H}$ is Lipschitz continuous with constant $\frac{\tau}{\gamma}$, i.e.,

$$\|R^{\mathcal{H},\eta}_{M,\lambda}(x) - R^{\mathcal{H},\eta}_{M,\lambda}(y)\| \leq \frac{\tau}{\gamma}\|x - y\|, \qquad \forall x, y \in H.$$

In this paper, we will prove a new existence result for problem (1.1) which is different from Theorem A.

## 2. Main result

In this section, we will prove the following new existence of solutions for problem (1.1).

**Theorem 2.1.** For $i = 1, 2$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be Lipshitz continuous with constant $\tau_i$, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly $\eta_i$-monotone and Lipschitz continuous with constant $\gamma_i$ and $\delta_i$, respectively, $g_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly monotone and Lipschitz continuous with constant $r_i$ and $s_i$ , respectively. Let $A, C : \mathcal{H}_1 \to CB(\mathcal{H}_1)$, $B, D : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be $\tilde{D}$-Lipschitz continuous with constants $l_A > 0, l_C > 0, l_B > 0$ and $l_D > 0$, respectively. Let $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ be strongly monotone with respect to $\hat{g}_1$ in the first argument with constant $\alpha_1 > 0$, Lipschitz continuous in the first argument with constant $\beta_1 > 0$, and Lipschitz continuous in the second argument with constant $\xi_1 > 0$, respectively, where $\hat{g}_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is defined by $\hat{g}_1(x) = H_1 \circ g_1(x) = H_1(g_1(x))$, $\forall x \in \mathcal{H}_1$. Let $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be strongly monotone with respect to $\hat{g}_2$ in the second argument with constant $\alpha_2 > 0$, Lipschitz continuous in the second argument with constant $\beta_2 > 0$, and Lipschitz continuous in the first argument with constant $\xi_2 > 0$, respectively, where $\hat{g}_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is defined by $\hat{g}_2(y) = H_2 \circ g_2(y) = H_2(g_2(y))$, $\forall y \in \mathcal{H}_2$. Assume that $P : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ is Lipschitz continuous in the first and second argument with constants $\mu_1 > 0$ and $\nu_1 > 0$, respectively, $Q : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ is Lipschitz continuous in the first and second argument with constants $\mu_2 > 0$ and $\nu_2 > 0$, respectively, $M : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is an $(H_1, \eta_1)$-monotone operator and $N : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is an $(H_2, \eta_2)$-monotone operator, respectively, $g_1^{-1}$ is Lipschitz continuous with constant $q$. If there exist constants $\lambda > 0$ and $\rho > 0$ such that

$$\theta_1 = [\sqrt{1 - 2r_2 + s_2^2} + \tfrac{\tau_2}{\gamma_2}(\sqrt{\delta_2^2 s_2^2 - 2\rho\alpha_2 + \rho^2\beta_2^2} + \rho\nu_2 l_D)]$$

$$\theta_2 = \frac{\lambda(\xi_1+\nu_1 l_B)q\frac{\tau_1}{\gamma_1}}{1-q\frac{\tau_1}{\gamma_1}(\sqrt{\delta_1^2 s_1^2-2\lambda\alpha_1+\lambda^2\beta_1^2}+\lambda\mu_1 l_A)}$$

$$\theta_3 = \rho(\xi_2 + \mu_2 l_C)\tfrac{\tau_2}{\gamma_2}$$

$$\theta = \theta_1 + \theta_2\theta_3 < 1 \tag{2.1}$$

then the problem (1.1) has a solution.

**Proof.** From Lemma 1.1, we only need to prove that $E(y)$ has a unique fixed point in $\mathcal{H}_2$. In fact, for any $y_1, y_2 \in \mathcal{H}_2$ and $y_1 \neq y_2$, if there exist $x_1, x_2 \in \mathcal{H}_1, u_1 \in A(x_1), v_1 \in B(y_1); u_2 \in A(x_2), v_2 \in B(y_2)$, satisfying

$$x_1 = g_1^{-1} R^{H_1,\eta_1}_{M,\lambda}[H_1(g_1(x_1)) - \lambda F(x_1, y_1) - \lambda P(u_1, v_1)] \tag{2.2}$$

$$x_2 = g_1^{-1} R^{H_1,\eta_1}_{M,\lambda}[H_1(g_1(x_2)) - \lambda F(x_2, y_2) - \lambda P(u_2, v_2)] \tag{2.3}$$

From definition of $E(y)$, we conclude that for any $t_1 \in E(y_1)$, $t_2 \in E(y_2)$, there exists $w_1 \in C(x_1), z_1 \in D(y_1); w_2 \in C(x_2), z_2 \in D(y_2)$, satisfying

$$t_1 = y_1-g_2(y_1)+R_{N,\rho}^{H_2,\eta_2}[H_2(g_2(y_1))-\rho G(x_1,y_1)-\rho Q(w_1,z_1)] \tag{2.4}$$

$$t_2 = y_2-g_2(y_2)+R_{N,\rho}^{H_2,\eta_2}[H_2(g_2(y_2))-\rho G(x_2,y_2)-\rho Q(w_2,z_2)] \tag{2.5}$$

Let $b_1 = H_2(g_2(y_1)) - \rho G(x_1, y_1) - \rho Q(w_1, z_1)$, $b_2 = H_2(g_2(y_2)) - \rho G(x_2, y_2) - \rho Q(w_2, z_2)$. From (2.4), (2.5) and Lemma 1.2, we have

$$\|t_1-t_2\| \le \|y_1-y_2-[g_2(y_1)-g_2(y_2)]\|+\tfrac{\tau_2}{\gamma_2}\|b_1-b_2\| \tag{2.6}$$

Since $g_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is strongly monotone and Lipschitz continuous with constant $r_2$ and $s_2$, respectively, we have

$$\begin{aligned} &\|y_1 - y_2 - [g_2(y_1) - g_2(y_2)]\|^2 \\ &\le \|y_1 - y_2\|^2 - 2\langle g_2(y_1) - g_2(y_2), y_1 - y_2\rangle + \|g_2(y_1) - g_2(y_2)\|^2 \\ &\le (1-2r_2+s_2^2)\|y_1-y_2\|^2 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \|b_1 - b_2\| &= \|H_2(g_2(y_1)) - \rho G(x_1, y_1) - \rho Q(w_1, z_1) \\ &\quad - [H_2(g_2(y_2)) - \rho G(x_2, y_2) - \rho Q(w_2, z_2)]\| \\ &\le \|H_2(g_2(y_1)) - H_2(g_2(y_2)) - \rho[G(x_1, y_1) - G(x_1, y_2)]\| \\ &\quad +\rho\|G(x_1,y_2)-G(x_2,y_2)\|+\rho\|Q(w_1,z_1)-Q(w_2,z_2)\| \end{aligned} \tag{2.8}$$

Since $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ is strongly monotone with respect to $\hat{g}_2 = H_2 \circ g_2$ in the second argument with constant $\alpha_2 > 0$ and Lipschitz continuous in the second argument with constant $\beta_2 > 0$, respectively, we obtain

$$\begin{aligned} &\|H_2(g_2(y_1)) - H_2(g_2(y_2)) - \rho[G(x_1, y_1) - G(x_1, y_2)]\|^2 \\ &\le \|H_2(g_2(y_1)) - H_2(g_2(y_2))\|^2 - 2\rho\langle G(x_1, y_1) - G(x_1, y_2), H_2(g_2(y_1)) - H_2(g_2(y_2))\rangle \\ &\quad + \rho^2\|G(x_1, y_1) - G(x_1, y_2)\|^2 \\ &\le (\delta_2^2 s_2^2-2\rho\alpha_2+\rho^2\beta_2^2)\|y_1-y_2\|^2 \end{aligned} \tag{2.9}$$

Since $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ is Lipschitz continuous in the first arguments with constant $\xi_2 > 0$, we have

$$\|G(x_1,y_2)-G(x_2,y_2)\| \le \xi_2\|x_1-x_2\| \tag{2.10}$$

It follows from the Lipschitz continuity of $Q$, the $\tilde{D}$-Lipschitz continuity of $C$ and $D$, we have that

$$\begin{aligned} &\|Q(w_1, z_1) - Q(w_2, z_2)\| \\ &\le \|Q(w_1, z_1) - Q(w_2, z_1)\| + \|Q(w_2, z_1) - Q(w_2, z_2)\| \\ &\le \mu_2\|w_1 - w_2\| + \nu_2\|z_1 - z_2\| \\ &\le \mu_2\tilde{D}(C(x_1), C(x_2)) + \nu_2\tilde{D}(D(y_1), D(y_2)) \\ &\le \mu_2 l_C\|x_1-x_2\|+\nu_2 l_D\|y_1-y_2\| \end{aligned} \tag{2.11}$$

By using (2.6)-(2.11), we have

$$\|t_1-t_2\| \le \theta_1\|y_1-y_2\|+\theta_3\|x_1-x_2\| \tag{2.12}$$

Where $\theta_1 = [\sqrt{1-2r_2+s_2^2} + \frac{\tau_2}{\gamma_2}(\sqrt{\delta_2^2 s_2^2 - 2\rho\alpha_2 + \rho^2\beta_2^2} + \rho\nu_2 l_D)]$

$\theta_3 = \rho(\xi_2 + \mu_2 l_C)\frac{\tau_2}{\gamma_2}$

Let $a_1 = H_1(g_1(x_1)) - \lambda F(x_1, y_1) - \lambda P(u_1, v_1)$, $a_2 = H_1(g_1(x_2)) - \lambda F(x_2, y_2) - \lambda P(u_2, v_2)$. In a similar way from (2.2) (2.3) and the Lipschitz continuity of $g_1^{-1}$, we have

$$\|x_1-x_2\| \le q\tfrac{\tau_1}{\gamma_1}\|a_1-a_2\| \tag{2.13}$$

and

$$\begin{aligned} \|a_1 - a_2\| &= \|H_1(g_1(x_1)) - \lambda F(x_1, y_1) - \lambda P(u_1, v_1) - [H_1(g_1(x_2)) - \lambda F(x_2, y_2) - \\ &\lambda P(u_2, v_2)]\| \\ &\le \|H_1(g_1(x_1)) - H_1(g_1(x_2)) - \lambda[F(x_1, y_1) - F(x_2, y_1)]\| \end{aligned}$$

$$+\lambda\|F(x_2,y_1)-F(x_2,y_2)\|+\lambda\|P(u_1,v_1)-P(u_2,v_2)\| \tag{2.14}$$

Since $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ is strongly monotone with respect to $\hat{g}_1 = H_1 \circ g_1$ in the first argument with constant $\alpha_1 > 0$ and Lipschitz continuous in the second argument with constant $\beta_1 > 0$, respectively, we obtain

$$\begin{aligned}
&\|H_1(g_1(x_1)) - H_1(g_1(x_2)) - \lambda[F(x_1,y_1) - F(x_2,y_1)]\|^2 \\
&\le \|H_1(g_1(x_1))-H_1(g_1(x_2))\|^2-2\lambda\langle F(x_1,y_1)-F(x_2,y_1), H_1(g_1(x_1))-H_1(g_1(x_2))\rangle \\
&\quad + \lambda^2\|F(x_1,y_1) - F(x_2,y_1)\|^2 \\
&\le (\delta_1^2 s_1^2 - 2\rho\alpha_1 + \rho^2\beta_1^2)\|x_1 - x_2\|^2
\end{aligned} \tag{2.15}$$

Since $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ is Lipschitz continuous in the second arguments with constant $\xi_1 > 0$, we have

$$\|F(x_2,y_1)-F(x_2,y_2)\| \le \xi_1\|y_1-y_2\| \tag{2.16}$$

It follows from the Lipschitz continuity of $P$, the $\tilde{D}$-Lipschitz continuity of $A$ and $B$, we have

$$\begin{aligned}
&\|P(u_1,v_1) - P(u_2,v_2)\| \\
&\le \|P(u_1,v_1) - P(u_2,v_1)\| + \|P(u_2,v_1) - P(u_2,v_2)\| \\
&\le \mu_1\|u_1 - u_2\| + \nu_1\|v_1 - v_2\| \\
&\le \mu_1\tilde{D}(A(x_1), A(x_2)) + \nu_1\tilde{D}(B(y_1), B(y_2)) \\
&\le \mu_1 l_A\|x_1-x_2\|+\nu_1 l_B\|y_1-y_2\|
\end{aligned} \tag{2.17}$$

It follows from (2.13)-(2.17), we conclude that

$$\begin{aligned}
\|x_1 - x_2\| &\le q\tfrac{\tau_1}{\gamma_1}(\sqrt{\delta_1^2 s_1^2 - 2\lambda\alpha_1 + \lambda^2\beta_1^2} + \lambda\mu_1 l_A)\|x_1 - x_2\| \\
&\quad +\lambda(\xi_1+\nu_1 l_B)q\tfrac{\tau_1}{\gamma_1}\|y_1-y_2\|]
\end{aligned} \tag{2.18}$$

It is easy to obtain that

$$\|x_1-x_2\| \le \frac{\lambda(\xi_1+\nu_1 l_B)q\frac{\tau_1}{\gamma_1}}{1-q\frac{\tau_1}{\gamma_1}(\sqrt{\delta_1^2 s_1^2-2\lambda\alpha_1+\lambda^2\beta_1^2}+\lambda\mu_1 l_A)}\|y_1-y_2\| \tag{2.19}$$

Let

$$\theta_2 = \frac{\lambda(\xi_1+\nu_1 l_B)q\frac{\tau_1}{\gamma_1}}{1-q\frac{\tau_1}{\gamma_1}(\sqrt{\delta_1^2 s_1^2-2\lambda\alpha_1+\lambda^2\beta_1^2}+\lambda\mu_1 l_A)}$$

It follows from (2.12) and (2.19) that

$$\|t_1-t_2\| \le \theta\|y_1-y_2\| \tag{2.20}$$

where $\theta = \theta_1 + \theta_2\theta_3$

From condition (2.1), we have $\theta < 1$ which indicate that $E(y)$ is a contraction mapping. By a fixed point Theorem of Nadler [3], we have for each $E(y)$ has a fixed point $y^*$, such that $y^* \in E(y^*)$. Then there exist $x^* \in \mathcal{H}_1, u^* \in A(x^*), v^* \in B(y^*), w^* \in C(x^*), z^* \in D(y^*), s.t.$

$$x^* = g_1^{-1}R_{M,\lambda}^{H_1,\eta_1}[H_1(g_1(x^*)) - \lambda F(x^*,y^*) - \lambda P(u^*,v^*)]$$

And

$$y^* = y^* - g_2(y^*) + R_{N,\rho}^{H_2,\eta_2}(H_2(g_2(y^*)) - \rho G(x^*,y^*) - \rho Q(w^*,z^*)]$$

By Lemma 1.1, we know that $(x^*,y^*)$ is a solution of problem (1.1).

This completes the proof of Theorem 2.1.

## References


1. J.W. Peng, D.L. Zhu, A new system of generalized mixed quasi-variational inclusions with $(H,\eta)$-monotone operators, J. Math. Anal. Appl. 327(2007), 175-187.
2. Y. P. Fan, N. J. Huang and H. B. Thompson, A new system of variational inclusions with $(H,\eta)$-monotone operators in Hilbert spaces, Computers Math. Appl. 49 (2005), 365-374.

3. S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30, 475-488, (1969).

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