Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/spm

(3s.) **v. 27** 1 (2009): 85–90.
ISSN-00378712 in press
doi:10.5269/bspm.v27i1.9070

Existence of solutions for a resonant Steklov Problem

Aomar ANANE, Omar CHAKRONE, Belhadj KARIM and Abdellah ZEROUALI

ABSTRACT: In this paper, we prove the existence of weak solutions to the problem $\triangle_p u = 0$ in Ω and $|\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = \lambda_1 m(x)|u|^{p-2}u + f(x,u) - h$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $m \in L^q(\partial\Omega)$ is a weight, λ_1 is the first positive eigenvalue for the eigenvalue Steklov problem $\triangle_p u = 0$ in Ω and $|\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = \lambda m(x)|u|^{p-2}u$ on $\partial\Omega$. f and h are functions that satisfy some conditions.

Key Words: : Steklov problem, Weights, Landesman-Lazer conditions, Palais–Smale conditions.

Contents

1. Introduction

Consider the problem

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = \lambda_1 m(x)|u|^{p-2}u + f(x,u) - h & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Ω will be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with a Lipschitz continuous boundary, $1 < p < \infty$, $m \in L^q(\partial\Omega)$ where $\frac{N-1}{p-1} < q < \infty$ if $p < N$ and $q \geq 1$ if $p \geq N$. We assume that $m^+ = \max(m, 0) \not\equiv 0$ and $\int_{\partial\Omega} m d\sigma < 0$. $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|f(x,s)| \leq a|s|^{r-1} + b(x) \tag{2}$$

for all $s \in \mathbb{R}$ and a.e. $x \in \partial\Omega$. Here $a = cst > 0$, $b \in L^{r'}(\partial\Omega)$ and $h \in L^{r'}(\partial\Omega)$, where r' is the conjugate of $r = \frac{pq}{q-1}$. λ_1 design the first positive eigenvalue of the following Steklov problem

$$\begin{cases} \text{To find } (u,\lambda) & \in & (W^{1,p}(\Omega) \setminus \{0\}) \times \mathbb{R}^+ & \text{such that} \\ \triangle_p u & = & 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} & = & \lambda m(x)|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{3}$$

It is well-known that

$$\lambda_1 := \inf_{u \in W^{1,p}(\Omega)} \left\{ \frac{1}{p}\int_\Omega |\nabla u|^p dx : \frac{1}{p}\int_{\partial\Omega} m(x)|u|^p d\sigma = 1 \right\}.$$

2000 *Mathematics Subject Classification:* 35J70, 35P30

Recall that λ_1 is simple (see [7]). Moreover, there exists a unique positive eigenfunction φ_1 whose norm in $W^{1,p}(\Omega)$ equals to one. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of (1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx = \lambda_1 \int_{\partial\Omega} m|u|^{p-2}u\varphi d\sigma + \int_{\partial\Omega} f(x,u)\varphi d\sigma - \int_{\partial\Omega} h\varphi d\sigma$$

for all $\varphi \in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure.

Classical Dirichlet problems involving the p-Laplacian have been studied by various authors, we cite the works [1], [2], [3], [4], [5], [6] and [8] . Our purpose of this paper is to extend some of the results known in the Dirichlet p-Laplacian problem. We prove the existence of solutions for a resonant Steklov problem under Landesman-Lazer conditions.

2. Existence of solutions for a resonant Steklov problem

In this section, we study the solvability of the Steklov problem (1) under Landesman-Lazer conditions and by using the minimum principle. The following theorem is our main ingredient.

Theorem 2.1 *(Minimum principle)*
Let X be a Banach space and $\Phi \in C^1(X,\mathbb{R})$. Assume that Φ satisfies the Palais–Smale condition and bounded from below. Then $c = \inf_X \Phi$ is a critical point.

Suppose that f satisfies the hypotheses below

$$\lim_{s\to-\infty} f(x,s) = l(x); \lim_{s\to+\infty} f(x,s) = k(x) \text{ a.e. } x \in \partial\Omega \tag{4}$$

$$\int_{\partial\Omega} k(x)\varphi_1 d\sigma < \int_{\partial\Omega} h(x)\varphi_1 d\sigma < \int_{\partial\Omega} l(x)\varphi_1 d\sigma, \tag{5}$$

where φ_1 is the normalized positive eigenfunction associated to λ_1.

The following theorem is main result in this paper.

Theorem 2.2 *Let $m \in L^q(\partial\Omega)$, $m^+ \neq 0$ and $\int_{\partial\Omega} m d\sigma < 0$. Assume (2), (4) and (5) are fulfilled. Then the problem (1) admits at least a weak solution in $W^{1,p}(\Omega)$.*

The following lemmas will be used in the proof of Theorem 2.2, it guarantees the existence of a critical point. The functional energy associated to the problem (1) is

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma - \int_{\partial\Omega} F(x,u)d\sigma + \int_{\partial\Omega} hu d\sigma,$$

where

$$F(x,t) := \int_0^t f(x,s)ds.$$

Lemma 2.1 *Let $m \in L^q(\partial\Omega)$, $m^+ \neq 0$ and $\int_{\partial\Omega} m d\sigma < 0$. Assume (4) and (5) are fulfilled. Then Φ satisfies the Palais–Smale condition (PS) on $W^{1,p}(\Omega)$.*

Proof: Let (u_n) be a sequences in $W^{1,p}(\Omega)$ and c be a real number such that $|\Phi(u_n)| \leq c$ for all n and $\Phi'(u_n) \to 0$. We prove that (u_n) is bounded in $W^{1,p}(\Omega)$, we assume by contradiction that $||u_n|| \to +\infty$ as $n \to +\infty$. Let $v_n = \frac{u_n}{||u_n||}$, thus v_n is bounded, for a subsequence still denoted by (v_n), we have $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$, $v_n \to v$ strongly in $L^p(\Omega)$ and $v_n \to v$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$. The hypothesis $|\Phi(u_n)| \leq c$ implies

$$\lim_{n\to+\infty} \left(\frac{1}{p} \int_\Omega |\nabla v_n|^p dx - \frac{\lambda_1}{p} \int_{\partial\Omega} m|v_n|^p d\sigma - \int_{\partial\Omega} \frac{F(x,u_n)}{||u_n||^p} d\sigma + \int_{\partial\Omega} h \frac{u_n}{||u_n||^p} d\sigma \right) = 0.$$

Since, by hypotheses on p, h, u_n and using (4)

$$\lim_{n\to+\infty} \left(- \int_{\partial\Omega} \frac{F(x,u_n)}{||u_n||^p} d\sigma + \int_{\partial\Omega} h \frac{u_n}{||u_n||^p} d\sigma \right) = 0,$$

while

$$\lim_{n\to+\infty} \frac{1}{p} \int_{\partial\Omega} m|v_n|^p d\sigma = \frac{1}{p} \int_{\partial\Omega} m|v|^p d\sigma,$$

we have

$$\lim_{n\to+\infty} \int_\Omega |\nabla v_n|^p dx = \lambda_1 \int_{\partial\Omega} m|v|^p d\sigma.$$

Using the weak lower semi-continuity of norm and the definition of λ_1, we get

$$\lambda_1 \int_{\partial\Omega} m|v|^p d\sigma \leq \int_\Omega |\nabla v|^p dx \leq \liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^p dx = \lambda_1 \int_{\partial\Omega} m|v|^p d\sigma.$$

Thus, $v_n \to v$ strongly in $W^{1,p}(\Omega)$ and

$$\lambda_1 \int_{\partial\Omega} m|v|^p d\sigma = \int_\Omega |\nabla v|^p dx.$$

This implies, by the definition of φ_1, that $v = \pm\varphi_1$ (since $\int_{\partial\Omega} m d\sigma < 0$).
Letting

$$g(x,s) = \begin{cases} \frac{F(x,s)}{s}, & \text{if } s \neq 0; \\ f(x,0), & \text{if } s = 0. \end{cases}$$

Case 1: Suppose that $v_n \to \varphi_1$, then we have $u_n(x) \to +\infty$ and

$$f(x,u_n(x)) \to k(x) \text{ a.e. } x \in \partial\Omega,$$

$$g(x,u_n(x)) \to k(x) \text{ a.e. } x \in \partial\Omega.$$

Therefore, the Lebesgue theorem implies that

$$\lim_{n+\infty} \int_{\partial\Omega} \left(pg(x,u_n(x)) - f(x,u_n(x))\right) v_n d\sigma = (p-1) \int_{\partial\Omega} k(x)\varphi_1(x) d\sigma.$$

On the other hand, $|\Phi(u_n)| \leq c$ implies that

$$-cp \leq \int_\Omega |\nabla v_n|^p dx - \lambda_1 \int_{\partial\Omega} m|v_n|^p d\sigma - \int_{\partial\Omega} pF(x,u_n)d\sigma + \int_{\partial\Omega} hu_n d\sigma \leq cp, \quad (6)$$

and $\Phi'(u_n) \to 0$ implies that for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$-\varepsilon \leq -\int_\Omega |\nabla u_n|^p dx + \lambda_1 \int_{\partial\Omega} m|u_n|^p d\sigma + \int_{\partial\Omega} f(x,u_n(x))u_n(x)d\sigma - \int_{\partial\Omega} h(x)u_n(x)d\sigma \leq \varepsilon. \quad (7)$$

By summing up (6) and (7), we get

$$\int_{\partial\Omega} f(x,u_n(x))u_n(x)d\sigma - \int_{\partial\Omega} pF(x,u_n)d\sigma + (p-1)\int_{\partial\Omega} h(x)u_n(x)d\sigma \geq -cp-\varepsilon,$$

dividing by $||u_n||$, we obtain

$$\int_{\partial\Omega} f(x,u_n(x))v_n(x)d\sigma - \int_{\partial\Omega} pg(x,u_n)v_n(x)d\sigma + (p-1)\int_{\partial\Omega} h(x)v_n(x)d\sigma \geq \frac{-cp-\varepsilon}{||u_n||}.$$

Passing to the limit, we obtain

$$\int_{\partial\Omega} h(x)\varphi_1(x)d\sigma \geq \int_{\partial\Omega} k(x)\varphi_1(x)d\sigma,$$

which contradicts (5).

Case 2: Suppose that $v_n \to -\varphi_1$, then we have $u_n(x) \to -\infty$ and

$$f(x,u_n(x)) \to l(x) \text{ a.e. } x \in \partial\Omega,$$

$$g(x,u_n(x)) \to l(x) \text{ a.e. } x \in \partial\Omega.$$

By summing up (6) and (7), we get

$$\int_{\partial\Omega} f(x,u_n(x))u_n(x)d\sigma - \int_{\partial\Omega} pF(x,u_n)d\sigma + (p-1)\int_{\partial\Omega} h(x)u_n(x)d\sigma \leq cp+\varepsilon,$$

dividing by $||u_n||$, we obtain

$$\int_{\partial\Omega} f(x,u_n(x))v_n(x)d\sigma - \int_{\partial\Omega} pg(x,u_n)v_n(x)d\sigma + (p-1)\int_{\partial\Omega} h(x)v_n(x)d\sigma \leq \frac{cp+\varepsilon}{||u_n||}.$$

Passing to the limits, we get

$$\int_{\partial\Omega} l(x)\varphi_1(x)d\sigma \leq \int_{\partial\Omega} h(x)\varphi_1(x)d\sigma.$$

which contradicts (5). Finally, (u_n) is bounded in $W^{1,p}(\Omega)$, for a subsequences still denoted by (u_n), there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$

and $u_n \to u$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$. By the hypotheses on m, h, u_n and using (4), we deduce that

$$\lim_{n\to+\infty}\int_{\partial\Omega} m|u_n|^{p-2}u_n(u_n-u)d\sigma = 0,$$

$$\lim_{n\to+\infty}\int_{\partial\Omega} f(x,u_n)(x)(u_n-u)d\sigma = 0,$$

$$\lim_{n\to+\infty}\int_{\partial\Omega} h(u_n-u)d\sigma = 0.$$

On the other hand, we have

$$\lim_{n\to+\infty}\Phi'(u_n)(u_n-u) = 0,$$

therefore

$$\lim_{n\to+\infty}\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)dx = 0,$$

of more $u_n \to u$ strongly in $L^p(\Omega)$, thus

$$\lim_{n\to+\infty}\int_{\Omega}|u_n|^{p-2}u_n(u_n-u)dx = 0,$$

it then follows from the (S^+) property that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. □

Lemma 2.2 *Let $m \in L^q(\partial\Omega)$, $m^+ \neq 0$ and $\int_{\partial\Omega} m d\sigma < 0$. Assume (4) and (5) are fulfilled. Then Φ is bounded from below.*

Proof: It suffices to show that Φ is coercive. Suppose by contradiction that there exists a sequence (u_n) such that $||u_n|| \to +\infty$ and $\Phi(u_n) \leq c$. As in proof of Lemma 2.1, we can show that $v_n = \frac{u_n}{||u_n||} \to \pm\varphi_1$. By the definition of λ_1, we have

$$0 \leq \int_{\Omega}|\nabla u_n|^p dx - \lambda_1\int_{\partial\Omega} m|u_n|^p d\sigma,$$

thus

$$-\int_{\partial\Omega} F(x,u_n(x))d\sigma + \int_{\partial\Omega} hu_n d\sigma \leq \Phi(u_n) \leq c. \tag{8}$$

Case 1: Suppose that $v_n \to \varphi_1$. Dividing (8) by $||u_n||$, we obtain

$$-\int_{\partial\Omega}\frac{F(x,u_n(x))}{||u_n||}d\sigma + \int_{\partial\Omega}\frac{hu_n}{||u_n||}d\sigma \leq \frac{\Phi(u_n)}{||u_n||} \leq \frac{c}{||u_n||}.$$

Passing to the limit, we get

$$-\int_{\partial\Omega} k(x)\varphi_1 d\sigma + \int_{\partial\Omega} h(x)\varphi_1 d\sigma \leq 0,$$

which contradicts (5).
Case 2: Assume that $v_n \to -\varphi_1$. Dividing (8) by $||u_n||$, we obtain

$$-\int_{\partial\Omega} \frac{F(x,u_n(x))}{||u_n||} d\sigma + \int_{\partial\Omega} \frac{hu_n}{||u_n||} d\sigma \le \frac{\Phi(u_n)}{||u_n||} \le \frac{c}{||u_n||}.$$

Passing to the limit, we get

$$\int_{\partial\Omega} l(x)\varphi_1 d\sigma - \int_{\partial\Omega} h(x)\varphi_1 d\sigma \le 0,$$

which contradicts (4). □

Proof: [Proof of Theorem 2.2] Assumption (2) implies that Φ in a C^1 functional on $W^{1,p}(\Omega)$. By Lemma 2.1, Φ satisfies the Palais–Smale condition and it is bounded from below by Lemma 2.2. To prove that Φ attains its proper infimum in $W^{1,p}(\Omega)$ (see Theorem 2.2). Finally the problem (1) admits a least a weak solution. □

References

1. A. Anane and J. P. Gossez; Strongly nonlinear elliptic problems near resonance: a variational approach, *Comm. Partial Diff. Eqns.* **15** (1990), 1141–1159.

2. D. Arcoya and L. Orsina; Landesman–Lazer conditions and quasilinear elliptic equations, *Nonlinear Analysis* **28** (1997), 1623–1632.

3. L. Boccardo, P. Drabek and M, Kucera; Landesman–Lazer conditions for strongly nonlinear boundary value problem, *Comment. Math. Univ. Carolinae* **30** (1989), 411–427.

4. D. M. Duc and N. T. Vu, Non-uniformly elliptic equations of p-Laplacian type, Nonlinear Anaysis, 61 (2005), 1483–1495.

5. P. De Napoli and M. C. Mariani, Mountain pass solutions to equation of p-Laplacian type, Nonlinear Anaysis, 54 (2003), 1205–1219.

6. Q. A. Ngo and H. Q. Toan, Existence of solutions for a resonant problem under Landesman-Lazer conditions, Electronic Journal of Differential Equations, Vol. 2008(2008), No. 98 and pp. 1–10.

7. O. Torné, Steklov problem with an indefinite weight for the p–Laplacien, Electronic Journal of Differential Equations, Vol. 2005(2005), No. 87 and pp. 1–8.

8. N. T. Vu, Mountain pass solutions and non-uniformly elliptic to equation, Vietnam J. of Math. 33:4 (2005), 391–408. Non-uniformly

Aomar ANANE –ananeomar@yahoo.fr
Omar CHAKRONE – chakrone@yahoo.fr
Belhadj KARIM – karembelf@hotmail.com
Abdellah ZEROUALI – abdellahzerouali@yahoo.fr
Université Mohamed I, Faculté des Sciences,
Département de Mathématiques et Informatique,
Oujda, Maroc